$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad T = ct \qquad \frac{\partial^2 u}{\partial T^2} = \frac{\partial^2 u}{\partial x^2}$$

$u : \mathbb{R}^2 \to \mathbb{R}$ a C^2 solution

$\Leftrightarrow$ $\exists$ functions $F, G : \mathbb{R} \to \mathbb{R}$

with

$u(x,t) = F(x+t) + G(x-t)$

d'Alembert's formula

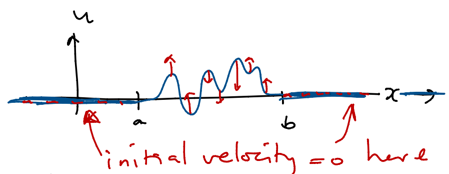
$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi$$

where $\left.\begin{array}{l} f(x) = u(x,0) \\ g(x) = u_t(x,0) \end{array}\right\}$ for all $x \in \mathbb{R}$

Finite speed of propagation.

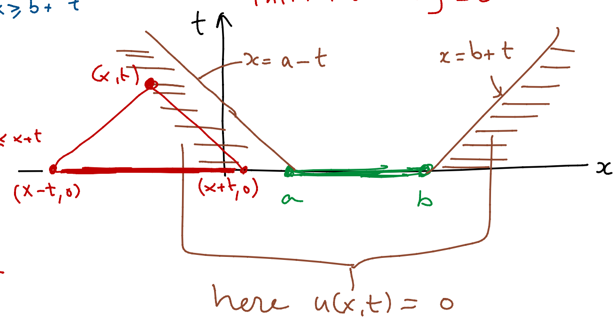
If $u(x,0) = u_t(x,0) = 0$ for all $x \in \mathbb{R}$ with $x \le a$ or $x \ge b$ then

$u(x,t) = 0$ when $x \le a - t$ or $x \ge b + t$



Proof Suppose

$$u(x,t) = \frac{\overbrace{u(x+t,0) + u(x-t,0)}^{=0}}{2} + \frac{1}{2}\underbrace{\int_{x-t}^{x+t} u_t(\xi, 0)\, d\xi}_{=0 \text{ for } x-t \le \xi \le x+t} = 0 \qquad ////$$



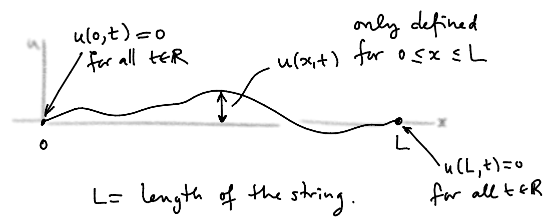
Boundary conditions.

How to solve

$$\begin{cases} u_{tt} = u_{xx} & \\ u(x,0) = f(x) & 0 \le x \le L \\ u_t(x,0) = g(x) & 0 \le x \le L \\ u(0,t) = u(L,t) = 0 & t \in \mathbb{R} \end{cases}$$

$\left.\begin{array}{l}0 \le x \le L\\ 0 \le x \le L\end{array}\right]$ initial conditions

$t \in \mathbb{R} \leftarrow$ boundary conditions



L = length of the string.

Trick: reflect the solution by

$u(-x,t) = -u(x,t)$ $\qquad (0 \le x \le L, \ \forall t)$

$\Rightarrow$ $u(-x,0) = -u(x,0)$ so $f(-x) = -f(x)$

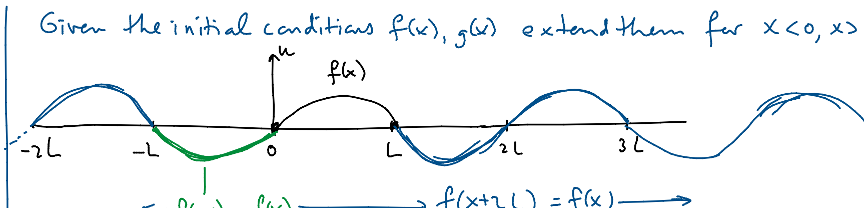
$u_t(-x,0) = -u_t(x,0)$ so $g(-x) = -g(x)$

$u(x+2L,t) = u(x,t)$ $\qquad (\forall x, t)$

$f(x+2L) = f(x)$

$g(x+2L) = g(x)$

Given the initial conditions $f(x)$, $g(x)$ extend them for $x<0$, $x>L$.



Then

$$u(x,t) = \frac{f(x+t) + f(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi$$

is defined for all $x \in \mathbb{R}$, $t \in \mathbb{R}$.

u satisfies
- the PDE ✓ (d'Alembert)
- the initial conditions (d'Alembert)

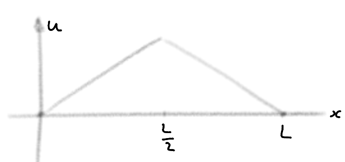
Does u satisfy the boundary conditions?

$$u(0,t) = \frac{\cancel{f(t) + f(-t)}^{=0}}{2} + \frac{1}{2}\int_{-t}^{t} g(\xi)\, d\xi = 0$$

$f(t) = -f(-t)$ $\qquad$ g odd $\quad \int_{-t}^{+t} g = 0$

$u(L,t) = 0$ D.I.Y.

Example The plucked string



$$u(x,0) = \begin{cases} x & 0 \le x \le \frac{L}{2} \\ L - x & \frac{L}{2} \le x \le L \end{cases}$$

$u_t(x,0) = 0$

Extend $u(x,0)$, $u_t(x,0)$