D'Alembert's solution

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \text{wave equation} \qquad (c=1)$$

d'Alembert: substitute $u(x,t) = v(x+t, x-t)$
find the function $v(r,s)$
Every function $u(x,t)$ can be written as $v(x+t, x-t)$: given u, define

$$(*) \quad v(r,s) \overset{\text{def}}{=} u\left(\frac{r+s}{2}, \frac{r-s}{2}\right) \qquad \text{for all } r,s$$

Then $v(x+t, x-t) = u(x,t)$ for all x,t
(verify by substituting $r = x+t$, $s = x-t$ in $(*)$)

We found:

Theorem If $u(x,t) = v(x+t, x-t)$ then

$$\boxed{\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for all } x,t} \iff \boxed{\frac{\partial^2 v}{\partial r \partial s} = 0 \quad \text{for all } r,s}$$

Theorem (a) If $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ for all x,t then there exist functions $F, G: \mathbb{R} \to \mathbb{R}$ such that

$$u(x,t) = F(x+t) + G(x-t)$$

(b) If $F, G: \mathbb{R} \to \mathbb{R}$ are C^2 functions, then $u(x,t) = F(x+t) + G(x-t)$ is a solution of the Wave Equation

Proof (a) done on Wednesday
(b) Verify by direct substitution:

$$\begin{aligned}
u(x,t) &= F(x+t) + G(x-t) \\
u_x &= F'(x+t) + G'(x-t) \\
u_{xx} &= F''(x+t) + G'(x-t) \\
u_t &= F'(x+t) - G'(x-t) \\
u_{tt} &= F''(x+t) + G''(x-t)
\end{aligned}
\quad \Rightarrow \quad u_{tt} = u_{xx} \ \checkmark$$

The initial value problem.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad u(x,t) = F(x+t) + G(x-t)$$

At $t=0$: $\quad u(x,0) = F(x) + G(x) \quad$ for all $x \in \mathbb{R}$.

Consider $\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial t}\left(F(x+t) + G(x-t)\right)$

$$= F'(x+t) - G'(x-t)$$

$$\underline{u_t(x,0) = F'(x) - G'(x)}$$

Also $\frac{\partial u}{\partial x}(x,t) = F'(x+t) + G'(x+t)$

$$\underline{u_x(x,0) = F'(x) + G'(x)}$$

$\Rightarrow$ Solve for $F'(x)$, $G'(x)$

Solution:

$$F'(x) = \frac{u_t(x,0) + u_x(x,0)}{2}$$

$$G'(x) = \frac{-u_t(x,0) + u_x(x,0)}{2}$$

Integrate to get F, G:

$$F(x) = F(0) + \int_0^x F'(\xi)\, d\xi$$

$$= F(0) + \frac{1}{2}\int_0^x \left(u_t(\xi,0) + u_x(\xi,0)\right) d\xi$$

$$\int_0^x u_x(\xi,0)\, d\xi = \int_0^x \frac{\partial u}{\partial x}(\xi,0)\, d\xi = u(x,0) - u(0,0)$$

$$F(x) = F(0) + \frac{1}{2}\int_0^x u_t(\xi,0)\, d\xi + \frac{1}{2}u(x,0) - \frac{1}{2}u(0,0)$$

Similar computation:

$$G(x) = G(0) - \frac{1}{2}\int_0^x u_t(\xi,0)\, d\xi + \frac{1}{2}u(x,0) - \frac{1}{2}u(0,0)$$

Therefore

$$u(x,t) = F(x+t) + G(x-t)$$

$$= F(0) + \underline{\frac{1}{2}\int_0^{x+t} u_t(\xi,0)\, d\xi} + \underline{\frac{1}{2}u(x+t,0)} - \frac{1}{2}u(0,0)$$

$$+ G(0) - \underline{\frac{1}{2}\int_0^{x-t} u_t(\xi,0)\, d\xi} + \underline{\frac{1}{2}u(x-t,0)} - \frac{1}{2}u(0,0)$$

$$= \frac{u(x+t,0) + u(x-t,0)}{2} + \frac{1}{2}\int_{x-t}^{x+t} u_t(\xi,0)\, d\xi + \underbrace{F(0) + G(0) - u(0,0)}_{=0}$$

(because $\int_0^{x+t} f(\xi)\, d\xi - \int_0^{x-t} f(\xi)\, d\xi = \int_0^{x+t} f(\xi)\, d\xi + \int_{x-t}^0 f(\xi)\, d\xi$

$= \int_{x-t}^{x+t} f(\xi)\, d\xi$)

Note: if $u(x,t) = F(x+t) + G(x-t)$ for all x,t,

then $u(0,0) = F(0) + G(0)$

so $F(0) + G(0) - u(0,0) = 0$

This gives us d'Alembert's formula for the solution:

$$u(x,t) = \frac{u(x+t,0) + u(x-t,0)}{2} + \frac{1}{2}\int_{x-t}^{x+t} u_t(\xi,0)\, d\xi \qquad (*)$$

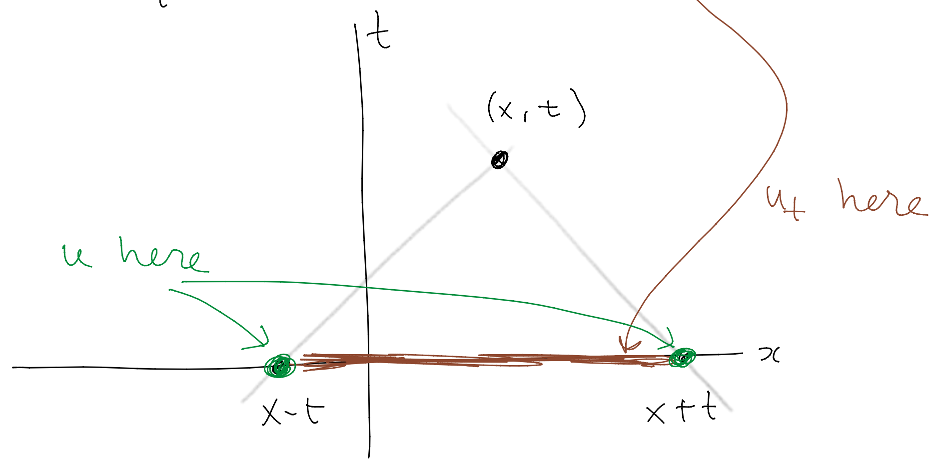
If we know $u(x,0)$ and $u_t(x,0)$ for all $x \in \mathbb{R}$

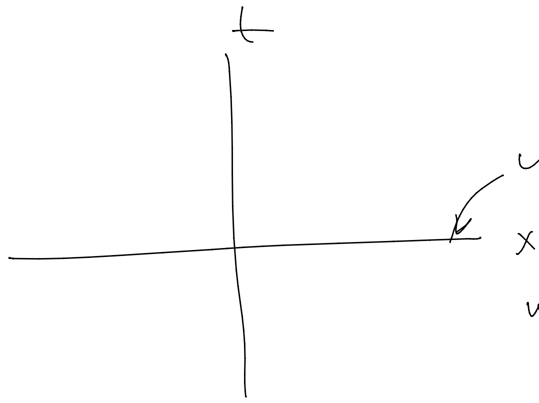
then the solution is given by $(*)$

More precisely, to know $u(x,t)$ we need to know

$\underline{u(x+t,0)}$ $\underline{u(x-t,0)}$

and $u_t(\xi,0)$ for $x-t \le \xi \le x+t$





$$u_t(x,0) = \begin{cases} x & x \ge 0 \\ x^2 & x \le 0 \end{cases}$$

Compute the solution:

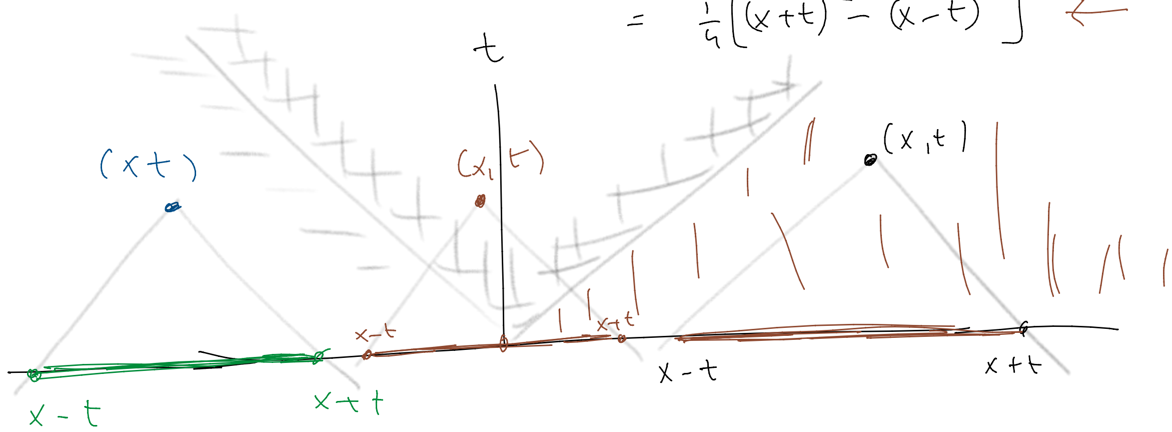
$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} u_t(\xi,0)\,d\xi$$

Assume $t > 0$.

If $x-t > 0$ (i.e. $x > t$) then $u_t(\xi,0) = \xi$ for $x-t < \xi < x+t$

and $u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \xi\,d\xi = \frac{1}{2}\left[\frac{1}{2}\xi^2\right]_{x-t}^{x+t}$

$$= \frac{1}{4}\left[(x+t)^2 - (x-t)^2\right]$$



If $x < -t$ then $x+t < 0$ and

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \xi^2\,d\xi = \frac{1}{6}\left[(x+t)^3 - (x-t)^3\right]$$

If $-t < x < t$ then

If $-t < x < t$ then

$$u(x,t) = \frac{1}{2} \int_{x-t}^{x+t} u_t(\xi, 0)\, d\xi$$

$$= \frac{1}{2} \left(\int_{x-t}^{0} \xi^2 \, d\xi + \int_{0}^{x+t} \xi \, d\xi \right)$$

$$= \ldots$$