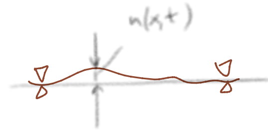
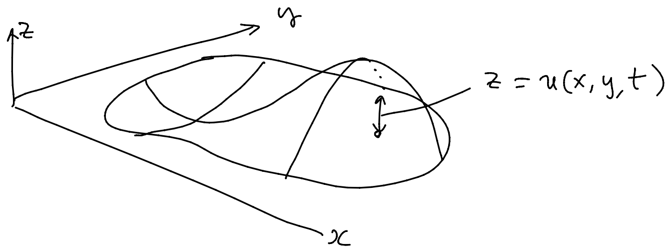
The wave equation

Vibrating string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$



Vibrating Membrane



$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Light waves

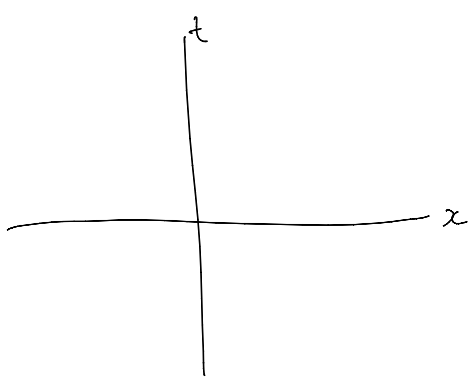
$\phi = \phi(x,y,z,t)$ = electric potential at $(x,y,z) \in \mathbb{R}^3$ at time t

Maxwell's equations $\Rightarrow$ $\dfrac{\partial^2 \phi}{\partial t^2} = c^2 \left(\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2} \right)$

Solution to 1-D wave equation according d'Alembert

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

assume $c = 1$



Idea: introduce new variables

$$r = x + t \qquad x = \frac{r+s}{2}$$
$$s = x - t \qquad t = \frac{r-s}{2}$$

$r + s = 2x \quad (+)$

$r - s = 2t \quad (-)$

Substitute in u:

$$v(r,s) = u(x,t) = u\left(\frac{r+s}{2}, \frac{r-s}{2} \right)$$

$$u(x,t) = v(r,s) = v(x+t,\, x-t)$$

If $u(x,t)$ satisfies $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ then what equation do we get for v?

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} v(\overbrace{x+t}^{r}, \overbrace{x-t}^{s})\right)$$

$$= \frac{\partial}{\partial t}\left[\frac{\partial v}{\partial r}\cdot \underbrace{\frac{\partial\, x+t}{\partial t}}_{=+1} + \frac{\partial v}{\partial s}\underbrace{\frac{\partial\, x-t}{\partial t}}_{=-1}\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{\partial v}{\partial r}(\underbrace{x+t}_{r}, \underbrace{x-t}_{s}) - \frac{\partial v}{\partial s}(\underbrace{x+t}_{r}, \underbrace{x-t}_{s})\right]$$

$$= \frac{\partial^2 v}{\partial r^2}(x+t, x-t)\cdot \overset{=+1}{\cancel{\frac{\partial\, x+t}{\partial t}}} + \frac{\partial^2 v}{\partial s \partial r}(x+t, x-t)\overset{=-1}{\left(\frac{\partial\, x-t}{\partial t}\right)}$$

$$- \frac{\partial^2 v}{\partial r \partial s}(x+t, x-t)\cdot \underset{=+1}{\left(\frac{\partial\, x+t}{\partial t}\right)} - \frac{\partial^2 v}{\partial s^2}(x+t, x-t)\overset{=-1}{\left(\frac{\partial\, x-t}{\partial t}\right)}$$

$$= v_{rr}(x+t, x-t) - \underbrace{v_{sr}(x+t, x-t) - v_{rs}(x+t, x-t)}_{v_{rs} = v_{sr}} + v_{ss}(x+t, x-t)$$

$$\frac{\partial^2 u}{\partial t^2} = v_{rr} - 2v_{sr} + v_{ss} \quad \text{at } (x+t, x-t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\left(v(\overbrace{x+t}^{r}, \overbrace{x-t}^{s})\right) = \frac{\partial}{\partial x}\left[v_r(\underbrace{x+t}_{r}, \underbrace{x-t}_{s}) + v_s(\underbrace{x+t}_{r}, \underbrace{x-t}_{s})\right]$$

$$= v_{rr}(x+t, x-t) + v_{rs}(x+t, x-t) + v_{sr}(x+t, x-t) + v_{ss}(x+t, x-t)$$

$$= v_{rr} + 2v_{rs} + v_{ss} \quad \text{at } (x+t,\, x-t)$$

Wave equation for $u \iff \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ for all (x,t)

$$\iff \cancel{v_{rr}} - 2v_{sr} + \cancel{v_{ss}} = \cancel{v_{rr}} + 2v_{sr} + \cancel{v_{ss}}$$

$$v_{rr}(x+t, x-t) - 2v_{sr}(..) + v_{ss}(..) = v_{rr}(..) + 2v_{sr}(..) + v_{ss}(..)$$

for all (x,t)

$(x,t) \longleftrightarrow (r,s)$ is bijective $\quad r = x+t,\ s = x-t \qquad x = \frac{1}{2}(r+s),\ t = \frac{1}{2}(r-s)$

$\Longleftrightarrow \quad 4 v_{sr}(r,s) = 0$ for all r, s

$\Longleftrightarrow \quad v_{sr}(r,s) = 0$ for all r, s

$\Longleftrightarrow \quad \dfrac{\partial^2 v}{\partial s \partial r}(r,s) = 0$ for all r, s

Solution of $\dfrac{\partial^2 v}{\partial s \partial r} = 0$:

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial v}{\partial r}\right) = 0$ for all $s, r \Longleftrightarrow \dfrac{\partial v}{\partial r}(s,r)$ does not depend on s.

$\Longleftrightarrow \quad \dfrac{\partial v}{\partial r}(s,r) = f(r)$

$$\Longleftrightarrow \quad v(s,r) = \underbrace{\int f(r)\,dr}_{F(r)} + G(s) = F(r) + G(s)$$

$u_{tt} = u_{xx}$ for all $(x,t) \Longleftrightarrow v_{rs} = 0$ for all r, s

$\Longleftrightarrow$ There exist functions $F, G : \mathbb{R} \to \mathbb{R}$ with $v(r,s) = F(r) + G(s)$

$\Longleftrightarrow \quad u(x,t) = v(x+t, x-t) = F(x+t) + G(x-t)$