Inviscid Burger's equation

Consider the so-called *inviscid Burger's equation*

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0,\qquad u(x,0)=u_0(x).$$

If $u(x,t)$ is a C^1 solution then any level set of u on which $u_x \neq 0$ is a straight line.

Suppose a level set is a graph $x = x_c(t)$, i.e. suppose that for some function $x = x_c(t)$ one has $u(x_c(t),t) = c$ for all t.

Proof: We show $x_c'(t) = c$.

Start from $u(x_c(t), t) = c$ for all t

Differentiate w.r.t. time, use $u_t = -u u_x$:

$$\frac{d}{dt}u(x_c(t),t) = \frac{d}{dt}c$$

↓ chain rule ↓ deriv of constant is 0

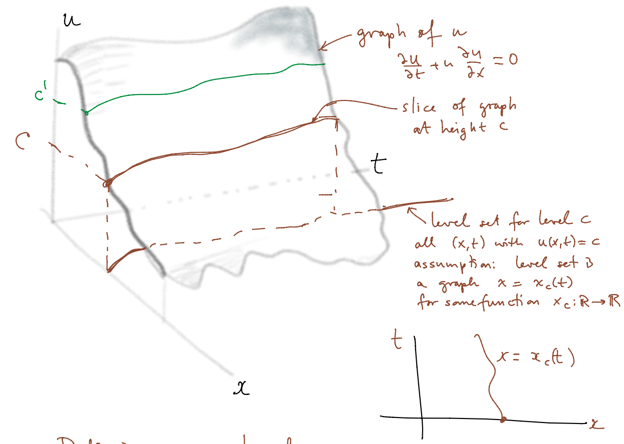
$$\frac{\partial u}{\partial x}(x_c,t)\cdot x_c'(t) + \frac{\partial u}{\partial t} = 0$$

↓ PDE

$$u_x \cdot x_c'(t) - u\cdot u_x = 0$$

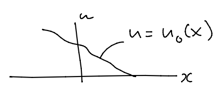
$$\underbrace{u_x(x_c,t)}_{\neq 0 \text{ by assumption}}\cdot\left(x_c'(t) - u(x_c,t)\right) = 0$$

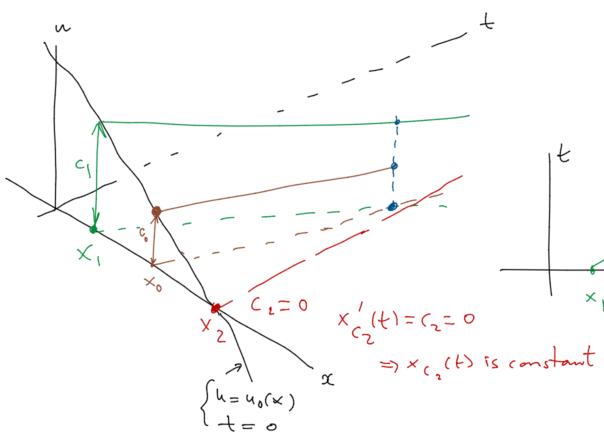
$x_c'(t) = u(x_c(t),t) = c$ by def of x_c ////

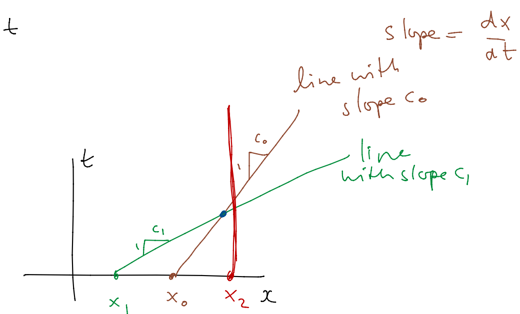


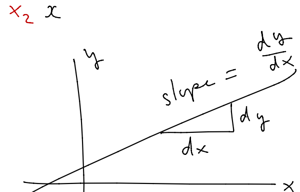
Defining property of x_c:
$u(x_c(t), t) = c$

Given $u(x,0) = u_0(x)$ for all x
can we construct the solution?









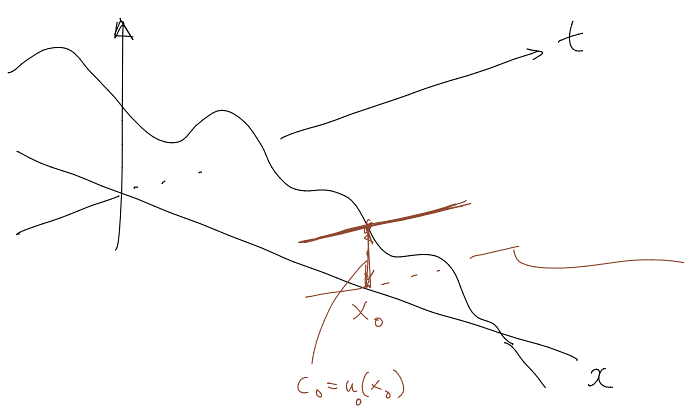
Theorem For every $M>0$ there is a $\delta>0$ such that
If $|u_0'(x)| \leq M$ for all $x\in\mathbb{R}$ then there exists a solution $u(x,t)$ of

$$u_t + u u_x = 0 \qquad x\in\mathbb{R},\ 0\leq t \leq \delta$$

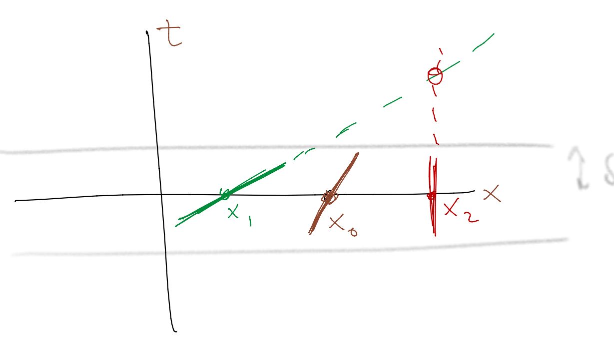
$$u(x,0) = u_0(x) \qquad x\in\mathbb{R}$$

Proof Let $x_0\in\mathbb{R}$ be given.
The levelset of u through $(x_0, 0)$ is the line

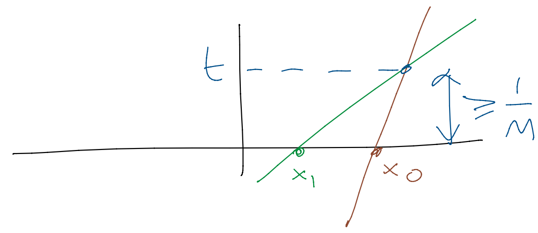
$$x = x_0 + c_0 t$$
$$= x_0 + u_0(x_0)\, t$$



Idea of proof: we show the line segments $x = x_0 + u_0(x_0)t$ ($|t|\leq\delta$) do not intersect.



Suppose $x_0 + u_0(x_0)\,t = x_1 + u_0(x_1)\,t$ where $x_0 \neq x_1$



Then

$$u_0(x_0)t - u_0(x_1)t = x_1 - x_0$$

$$(u_0(x_0) - u_0(x_1))\,t = x_1 - x_0 \neq 0 \quad \text{(by assumption } x_0 \neq x_1)$$

$$\frac{u_0(x_0) - u_0(x_1)}{x_1 - x_0} \cdot t = 1$$

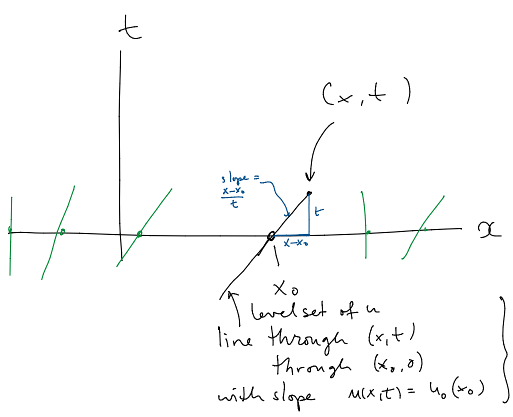
$$-\frac{u_0(x_1) - u_0(x_0)}{x_1 - x_0} \cdot t = 1$$

Mean Value Theorem: There is $\xi \in (x_0, x_1)$ with $\frac{u_0(x_1) - u_0(x_0)}{x_1 - x_0} = u_0'(\xi)$

$$\Rightarrow \quad -u_0'(\xi) \cdot t = 1$$

$$\Rightarrow \quad t = \frac{-1}{u_0'(\xi)}$$

Since $|u_0'(\xi)| \leq M$ we find $|t| = \frac{1}{|u_0'(\xi)|} \geq \frac{1}{M}$



Construction of a solution.

Q: Given (x,t) what is $u(x,t)$?

A: $u(x,t) = u(x_0, 0) = u_0(x_0)$

if

$x = x_0 + \text{slope} \cdot t$

$\boxed{x = x_0 + u(x_0)\,t}$

If $|t| \leq \delta = \frac{1}{2M}$ then $x = x_0 + u(x_0)t$ has exactly one solution $x_0 \in \mathbb{R}$ for each (x,t)

Proof. Let $t \in [-\delta, +\delta]$ and $x \in \mathbb{R}$ be given.

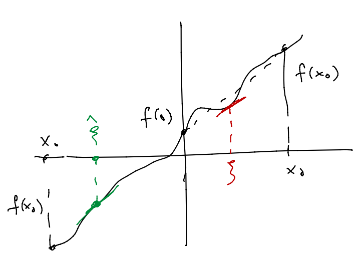
Consider $f(x_0) = x_0 + u(x_0)\,t$

Then $f'(x_0) = \frac{d\; x_0 + u_0(x_0)t}{dx_0} = 1 + u_0'(x_0)\,t$

$$\Rightarrow \quad |f'(x_0) - 1| = |u_0'(x)|\,|t| \leq M \cdot \frac{1}{2M} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2} \leq f'(x_0) \leq \frac{3}{2}.$$

We'll show: $\lim_{x_0 \to \pm\infty} f(x_0) = \pm\infty$



For $x_0 > 0$:

$$\frac{f(x_0) - f(0)}{x_0} = f'(\xi) \geq \frac{1}{2}$$

by the Mean Value Theorem.

$$\Rightarrow \quad f(x_0) \geq f(0) + \frac{x_0}{2}$$

$$\Rightarrow \quad \lim_{x_0 \to \infty} f(x_0) = +\infty$$

For $x_0 < 0$:

$$\frac{f(x_0) - f(0)}{x_0} = f'(\xi) \geq \frac{1}{2}$$

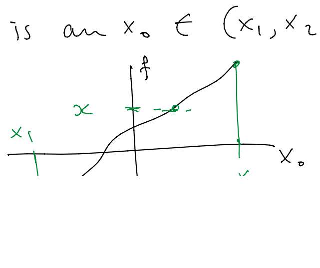
$$f(x_0) - f(0) \leq \frac{1}{2} x_0$$

$$f(x_0) \leq f(0) + \frac{1}{2} x_0 \quad (\forall x_0 < 0)$$

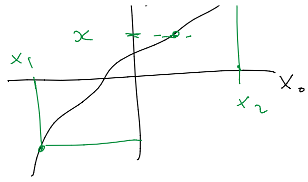
$$\Rightarrow \quad \lim_{x_0 \to -\infty} f(x_0) = -\infty.$$

Given $x \in \mathbb{R}$ choose $x_1 < 0$ with $f(x_1) < x$
$x_2 > 0$ with $f(x_2) > x$

The Intermediate Value Theorem implies that there is an $x_0 \in (x_1, x_2)$ with $f(x_0) = x$



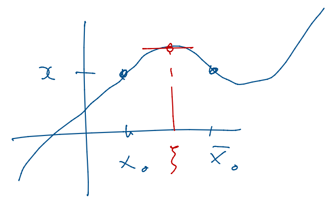
($f(x_0) \overset{\text{def}}{=} x_0 + t\,u_0(x_0)$ is a continuous function)



($f(x_0) = x_0 + u_0(x_0)\,t$ is a continuous function)

If $x_0 \neq \bar{x}_0$ both satisfy $f(x_0) = x$, $f(\bar{x}_0) = x$

then $0 = \dfrac{f(x_0) - f(\bar{x}_0)}{x_0 - \bar{x}_0} \overset{MVT}{=} f'(\xi)$



This contradicts " $f'(x_0) \geq \frac{1}{2}$ for all x_0 "

Now we can define $u(x,t)$ for $x \in \mathbb{R}$, $|t| \leq \frac{1}{2M} = \delta$ as follows:

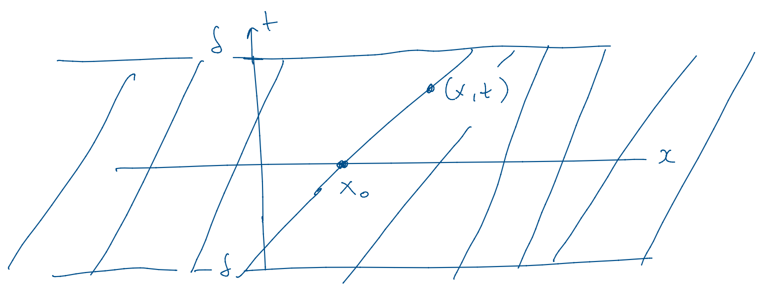
given (x,t) let $x_0 = X_0(x,t)$ be the unique solution of

$$\underline{x_0 + u_0(x_0)\,t = x}$$

(if $u_0(x_0) = \dfrac{1}{\cosh x_0}$ or $\sin(x_0)$ . .

then you have to solve $x_0 + \dfrac{t}{\cosh x_0} = x$ for x_0)

We define $u(x,t) = u_0(X_0(x,t))$



Claim $u(x,t)$ is continuously differentiable. (C^1)

If that is true then the fact that u is constant along the lines $x = x_0 + u(x_0)\,t$

implies $u_t + u u_x = 0$

namely: $u(x_0 + u(x_0)t,\ t) = u_0(x_0)$ for all $|t| \leq \delta$

$$\Rightarrow \frac{d}{dt} u(- - -) = 0$$

chainrule $\Rightarrow$

$$\frac{\partial u}{\partial x}(\underbrace{x_0 + u(x_0)t}_{=x},\ t)\cdot \underbrace{u(x_0)}_{=u(x,t)} + \frac{\partial u}{\partial t}(\underbrace{x_0 + u_0(x_0)t}_{x}, t) = 0$$

$$\Rightarrow u u_x + u_t = 0 \quad \text{at } (x,t).$$

Proof of the claim $x_0 = X_0(x,t)$ is the solution of

$$x_0 + u(x_0)\,t = x \qquad \text{i.e.} \qquad F(x_0, t, x) = 0$$

where $F(x_0, t, x) \overset{def}{=} \underline{x_0 + u(x_0)\,t - x}$

The IMPLICIT FUNCTION THEOREM says if

$$F(X_0(x,t), t, x) = 0 \quad \text{for all } x,t$$

and $\dfrac{\partial F}{\partial x_0}(x_0, t, x) \neq 0$ ⊛

then $X_0(x,t)$ is a C^1 of (x,t)

To verify ⊛ : $|t| \leq \delta$ $|u_0'(..)| \leq M$

$|t| \le \delta \quad |u_0'(\cdot)| \le M$

To verify ⊛ :

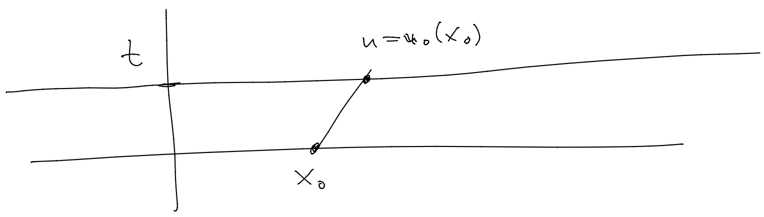
$$\frac{\partial}{\partial X_0} F(X_0, x, t) = \frac{\partial\, X_0 + t u_0(X_0) - x}{\partial X_0} = 1 + t u_0'(X_0)$$

$$\ge 1 - \delta \cdot M$$

$$= \frac{1}{2} \qquad \left(\delta = \frac{1}{2M}\right) \quad ✓$$

To graph the solutions consider this:

$$u(x,t) = u_0(X_0(x,t)) \qquad X_0 + t\, u_0(X_0) = x$$



The point $(x, u(x,t)) = (x_0 + t u_0(x_0), u(x,t))$

$$= (x_0 + t u_0(x_0), u_0(x_0))$$

lies on the graph of the solution.