Characteristics examples

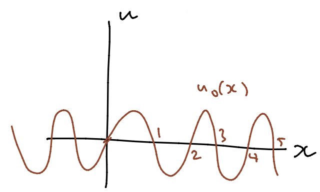
$$u = u(x,t)$$

Solve: $u_t - 4u_x = 2u, \quad u(x,0) = \sin(\pi x)$

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = \underline{f(x,t,u)}$$

$$a(x,t) = -4$$

$$f(x,t,u) = 2u$$

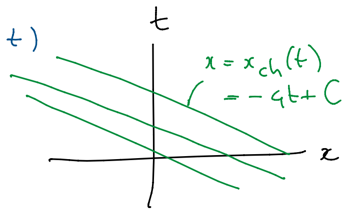


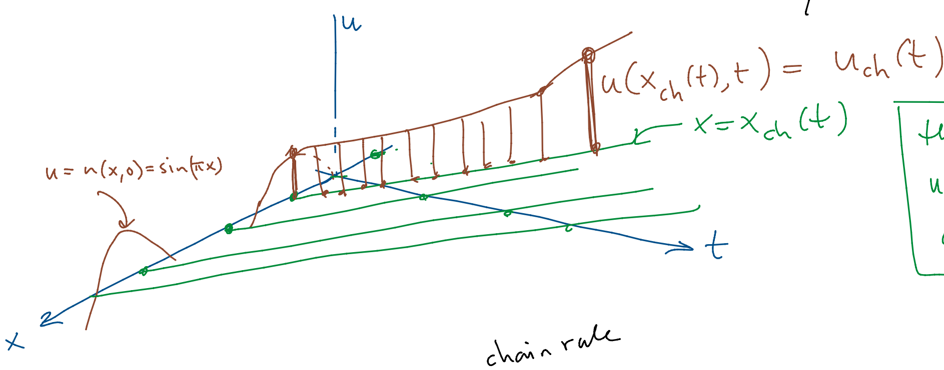
Method of characteristics.

A characteristic is any solution of $x_{ch}'(t) = a(x_{ch}(t), t)$

In this example $x_{ch}'(t) = -4$

solution: $x_{ch}(t) = -4t + C$





the PDE

$u_t - 4u_x = 2u$

$u_t = 4u_x + 2u$

$$\frac{du_{ch}(t)}{dt} = \frac{d\,u(x_{ch}(t), t)}{dt} \overset{\text{chain rule}}{=} \frac{\partial u}{\partial x}(x_{ch}(t), t)\cdot \underbrace{x_{ch}'(t)}_{=-4} + \underbrace{\frac{\partial u}{\partial t}(x_{ch}(t), t)}_{=u_t = 4u_x + 2u}\cdot \overset{\frac{dt}{dt}}{1}$$

$$= \cancel{u_x\cdot(-4)} + \cancel{4u_x} + 2u$$

$$= 2u(x_{ch}(t), t) \quad \Rightarrow \quad \boxed{\frac{du_{ch}}{dt} = 2u_{ch}}$$

$$= 2u_{ch}(t)$$

$$= f(x_{ch}(t), t, u_{ch}(t)) \quad \Rightarrow \quad u_{ch}(t) = e^{2t}\,u_{ch}(0)$$

Conclusion

$$u_t - 4u_x = 2u, \quad u(x,0) = \sin(\pi x)$$

$$x_{ch}(t) = -4t + C$$

$$u(x_{ch}(t), t) = e^{2t}\,u(x_{ch}(0), 0)$$

$$= e^{2t}\sin(\pi x_{ch}(0))$$

To find $u(\bar x,\bar t)$ choose C so that $x_{ch}(\bar t) = \bar x$

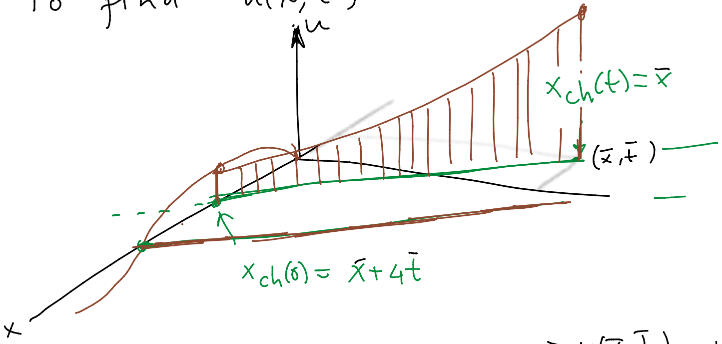
i.e. $-4\bar t + C = \bar x$

$C = \bar x + 4\bar t$

Then $x_{ch}(0) = -4\cdot 0 + C$

$= C$

$= \bar x + 4\bar t$



$\Rightarrow u(\bar x,\bar t) = u(x_{ch}(\bar t),\bar t = {}^{\bar t}\sin(\pi x_{ch}(0))$

$= e^{2\bar t}\sin\pi(\bar x + 4\bar t)$

for any $(\bar x,\bar t)\in\mathbb{R}^2$

$\Rightarrow$ The solution is $\underline{u(x,t) = e^{2t}\sin\pi(x+4t)}$

Interpretation

$$\frac{\partial u}{\partial t}\underbrace{-4\frac{\partial u}{\partial x}}_{\text{transport with velocity } -4} = \underbrace{2u}_{\text{exponential growth during transport}}$$