Inviscid Burger's equation

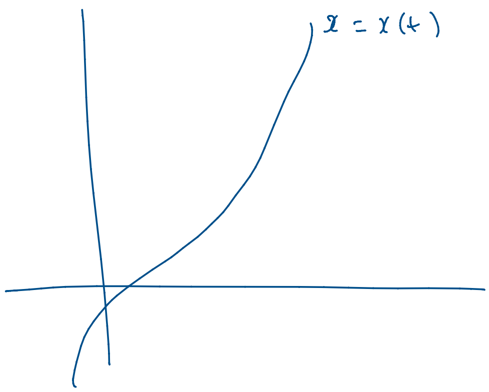
A nonlinear equation

Consider the so-called *inviscid Burger's equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

If $u(x,t)$ is a C^1 solution then any level set of u on which $u_x \neq 0$ is a straight line.

Suppose a level set is a graph $x = \overset{x}{f}(t)$, i.e. suppose that for some function $x = f(t)$ one has $u(\overset{x}{f}(t), t) = c$ for all t. Then . . .



suppose $x = x(t)$ is a level set of the solution $u(x,t)$, i.e.

$$u(x(t),t) = u(x(0),0) = c = u_0$$

for all $t \in \mathbb{R}$

$$\frac{d}{dt}\Big(u(x(t),t)\Big) = 0$$

because $u(x(t),t) = c$ for all t

Chain Rule:

$$u_x(x(t),t)\cdot x'(t) + u_t(x(t),t)\cdot \underbrace{\cancel{\frac{dt}{dt}}}_{=1} = 0$$

$u_t + u u_x = 0$

The PDE for u implies $u_t = -u u_x$, so

$$u_x \cdot x'(t) - u \cdot u_x = 0$$

Assume $u_x \neq 0$. Then divide by $u_x(x(t),t)$:

$$x'(t) - \overbrace{u(x(t),t)}^{=c} = 0$$

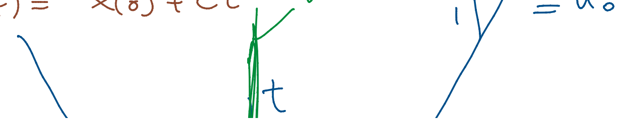
$$\Rightarrow \quad x'(t) = c$$

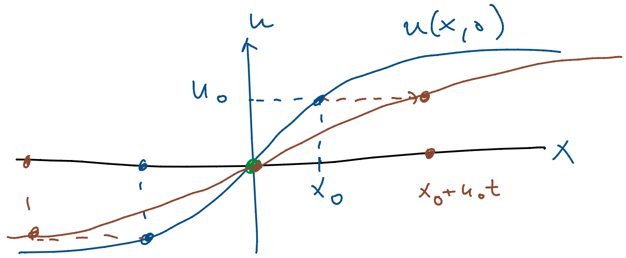
Conclusion: the levelset $u(x,t) = c$ is a straight line $x(t) = x(0) + ct$

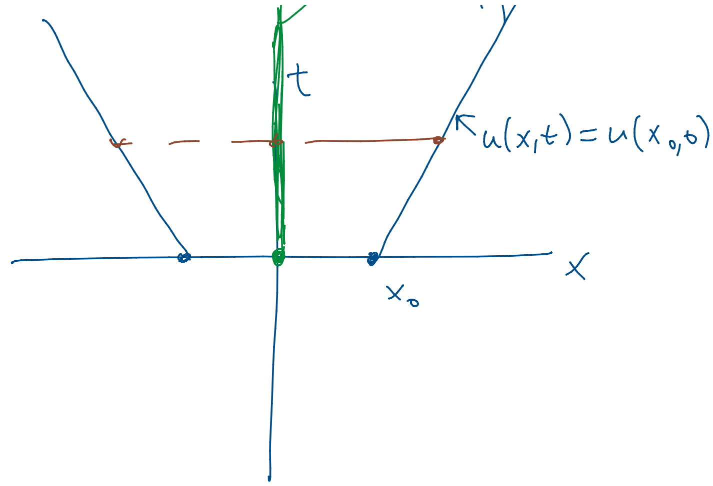
$u(x,t) = 0$

u_0 slope $= \frac{dx}{dt} = u_0$

1




u
u(x,0)
u_0
x
x_0
x_0+u_0t


t
$u(x,t)=u(x_0,0)$
x
x_0