Method of Characteristics

There is a general method, called the *method of characteristics* for finding the solutions to a first order equation of the form

$u_t + (x+t)\,u_x = \sin(u + e^t x)$

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = f(x,t,u)$$

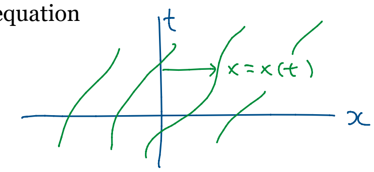
characteristics are solutions of $\frac{dx}{dt} = x+t$

in which $a\colon \mathbb{R}^2 \to \mathbb{R}$ and $f\colon \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable functions.

In this method we first find the characteristics of the equation. These are curves in the (x,t)–plane that are graphs of functions $x\colon \mathbb{R} \to \mathbb{R}$ that satisfy the characteristic equation

$$\frac{dx(t)}{dt} = a(x(t),t)$$

$x'(t) = a(x(t),t)$



If $x\colon \mathbb{R} \to \mathbb{R}$ is a characteristic of the PDE (1), i.e. if $x\colon \mathbb{R} \to \mathbb{R}$ satisfies (2), then the several variable chain rule implies

$$\frac{du(x(t),t)}{dt} = u_x(x(t),t)\cdot x'(t) + u_t(x(t),t).\ \underbrace{\frac{dt}{dt}}_{=1}$$

$u_t + a(x,t)u_x = f(x,t,u)$

$u_t = -a\cdot u_x + f$

Since u satisfies (1), we can write this as

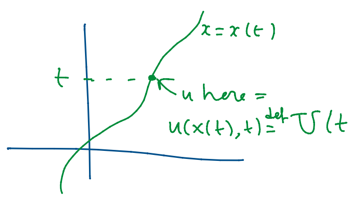
$$\frac{du(x(t),t)}{dt} = \cancel{u_x\cdot x'(t)} - \cancel{a(x(t),t)\cdot u_x} + f(x(t),t,u(x(t),t))$$

The characteristic equation (2) causes the first two terms on the left to cancel so that we find that *if* $x\colon \mathbb{R} \to \mathbb{R}$ *is a characteristic* then

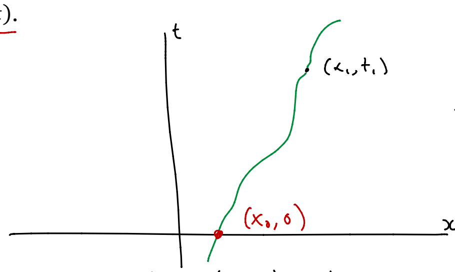
$$\frac{du(x(t),t)}{dt} = f(x(t),t,u(x(t),t))$$

$U(t) = u(x(t),t)$

$$\frac{d\,U(t)}{dt} = f(x(t),t,U(t))$$



This is an ordinary differential equation for the values of the solution u evaluated along the characteristic $x(t)$, i.e. for the function $t \mapsto u(x(t),t)$.



$u_t + a(x,t)u_x = f(x,t,u)$

Compute $u(x_1,t_1)$

Assume $u(x,0)$ given for all $x \in \mathbb{R}$

1. Find the characteristic through (x_1,t_1)
2. Find where the characteristic intersects the x-axis (at $(x_0,0)$)
3. Solve $\begin{cases} \frac{dU}{dt} = f(x(t),t,U(t)) & 0 \le t \le t_1 \\ U(0) = u(x_0,0) \end{cases}$
4. $u(x_1,t_1) = U(t_1)$

Transport with variable speed-example

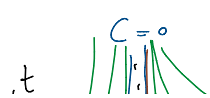
Let u be a solution of

$a(x,t) = x(1-x)$

$f(x,t,u) = 0$

$u_t + x(1-x)u_x = 0$

$t = \ln\left|\frac{x}{1-x}\right| + C$



Let u be a solution of

$a(x,t) = x(1-x)$

$f(x,t,u) = 0$

$$u_t + x(1-x)u_x = 0$$

The equation for characteristics is

$$\frac{dx}{dt} = a(x,t) = x(1-x)$$

Solve the diffeq: $t = \int \frac{dx}{x(1-x)} = \ln\left|\frac{x}{1-x}\right| + C$

So the characteristics are given by

$$t = \ln\left|\frac{x}{1-x}\right| + C$$

which you can also write as $x(t) = \frac{e^{t+C}}{e^{t+C} \pm 1}$

Given a point (x,t) the value of C for the characteristic through this point is

$$C = t - \ln\left|\frac{x}{1-x}\right|$$

The x-coordinate of the point where the characteristic intersects the x-axis satisfies

$$C = 0 - \ln\left|\frac{x_0}{1-x_0}\right|$$

Solve for x_0:

$$x_0 = \frac{xe^{-t}}{xe^{-t} + 1 - x}$$

The solution at (x,t) is therefore

$$u(x,t) = u(x_0, 0) = u\left(\frac{xe^{-t}}{xe^{-t} + 1 - x}, 0\right)$$