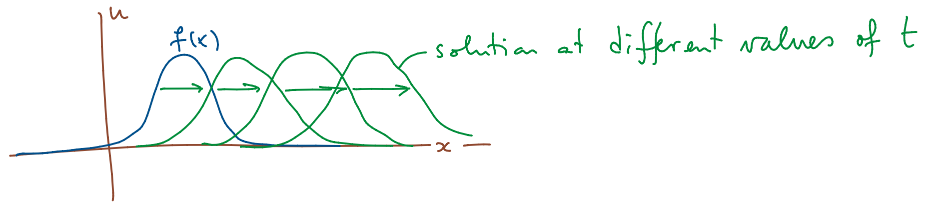
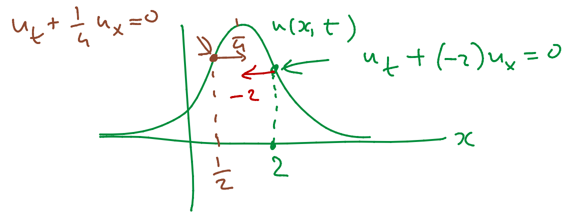
Transport equations: Using calculus

$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad c \in \mathbb{R}$ (a "constant")

Solution : $\quad u(x,t) = f(x - ct)$ where $f(x) = u(x,0)$



Transport with variable speed



If u is a solution of

$u_t + x(1-x)u_x = 0$

then can we find a function $x = x(t)$ such that $\frac{d}{dt}u(x(t),t) = 0$?

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0$$

$$a(x) = x(1-x)$$

$$\frac{d\,u(x(t),t)}{dt} \overset{\text{chainrule}}{=} \frac{\partial u}{\partial x}(x(t),t)\cdot\frac{dx(t)}{dt} + \frac{\partial u}{\partial t}(x(t),t)\underbrace{\frac{dt}{dt}}_{=1}$$

$= u_x \cdot x'(t) + u_t \qquad$ use $\quad u_t + x(1-x)u_x = 0$

$= u_x \cdot x'(t) - x(1-x)u_x \qquad$ ordinary diff eq. ODE for $x(t)$

$= u_x \cdot \left(x'(t) - x(1-x)\right)$

If we require $\boxed{\frac{dx}{dt} - x(1-x) = 0}$ then $\frac{d\,u(x(t),t)}{dt} = 0$

i.e. $\quad u(x(t),t) = u(x(0),0)$ does not depend on t.

Solving the ODE $\quad \frac{dx}{dt} = x(1-x) \quad$ "separable"

$$\int \frac{dx}{x(1-x)} = \int dt = t + C$$

$$\frac{1}{x(1-x)} = \frac{1-x+x}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x} \Rightarrow \int\frac{dx}{x(1-x)} = \ln x - \ln(1-x) = \ln\frac{x}{1-x}$$

$$\frac{x}{1-x} = e^t e^C$$

$$\frac{dx}{dt} = x(1-x) \rightarrow t + C = \int\frac{dx}{x(1-x)} = \ln\frac{x}{1-x} \Rightarrow \frac{x}{1-x} = e^{t+C} \Rightarrow x(t) = \frac{e^{t+C}}{e^{t+C}+1}$$

$x_0 = x(0)$

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Conclusion: if $u(x,t)$ satisfies $u_t + x(1-x)u_x = 0$

then $u\left(\frac{e^{t+C}}{e^{t+C}+1}, t\right) = u\left(\frac{e^C}{e^C+1}, 0\right)$

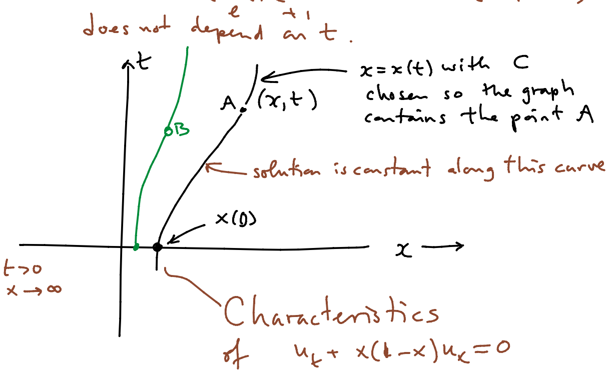
does not depend on t.

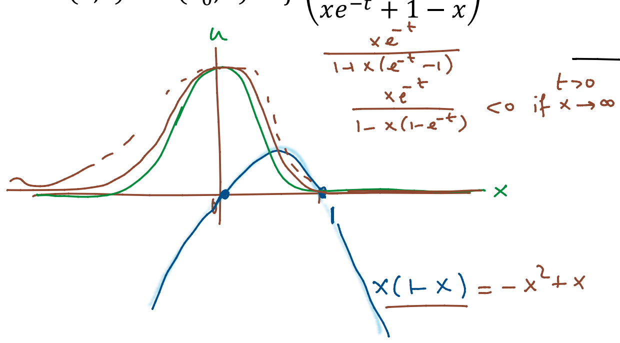
$$e^C = \frac{xe^{-t}}{1-x} \Rightarrow x_0 = \frac{e^C}{e^C+1} = \frac{xe^{-t}}{xe^{-t}+1-x}$$

$$u(x,t) = u(x_0,0) = f\left(\frac{xe^{-t}}{xe^{-t}+1-x}\right)$$

$$\frac{xe^{-t}}{1+x(e^{-t}-1)}$$

$$\frac{xe^{-t}}{1-x(1-e^{-t})} < 0 \text{ if } x \to \infty \quad t > 0$$





$$\frac{\partial u}{\partial t} + \underbrace{x(1-x)}\frac{\partial u}{\partial x} = 0$$

Equations for the derivatives of a solution

If u is a solution of

$$u_t + \sin(x)\,u_x = u^2$$

then the derivative $v = u_x$ also satisfies a partial differential equation.

We get this equation by differentiating the equation for u with respect to x on both sides:

$u_t + \sin(x)\, v = u^2$

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$$u_t + \sin(x)\, v = u^2$$

"Differentiate w.r.t. x on both sides:"

$$\frac{\partial\left(u_t + \sin(x)\, u_x\right)}{\partial x} = \frac{\partial u^2}{\partial x}$$

$$\underbrace{\frac{\partial u_t}{\partial x}}_{=v_t} + \cos(x)\, \underbrace{u_x}_{=v} + \sin(x)\, \underbrace{\frac{\partial \overset{=v}{u_x}}{\partial x}}_{=\frac{\partial v}{\partial x}} = 2u \cdot \underbrace{\frac{\partial u}{\partial x}}_{=v}$$

$$\frac{\partial u_t}{\partial x} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial v}{\partial t} = v_t$$

"mixed partials are equal"
(if they exist and are continuous)

If $u_t + \sin(x) u_x = u^2$ and if $v = \frac{\partial u}{\partial x}$ then

$$\Rightarrow \quad v_t + \cos(x)\, v + \sin(x)\, v_x = 2u\, v$$

$$v_t + \sin(x)\, v_x = 2u \cdot v - \cos(x)\, v = (2u - \cos x)\, v \qquad (*)$$

Next: $p = \frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x}$. Equation for p follows by differentiating $(*)$ w.r.t. x