The transport equation

The constant velocity transport equation — characteristics

We consider the equation $\frac{\partial u}{\partial t} = u_t = u_t(x,t)$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \qquad u_t(x,t) + c\, u_x(x,t) = 0$$

$u(x,t) = x - t$ solution? $\quad u_t = \frac{\partial x - t}{\partial t} = -1$

$u_x = \frac{\partial (x-t)}{\partial x} = +1$

$u_t + cu_x = -1 + c \cdot 1 = c - 1$

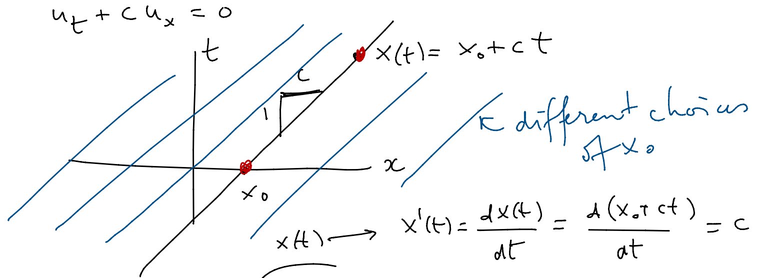
solution no unless $c = 1$

If $u(x,t)$ is a continuously differentiable solution, then by the several variable chain rule

$$\frac{du(x_0+ct,t)}{dt} = \frac{\partial u}{\partial x}\frac{d(x_0+ct)}{dt} + \frac{\partial u}{\partial t} = cu_x + u_t = 0$$

for any $x_o, t \in \mathbb{R}$. This means that any solution is constant along the lines $x(t) = x_0 + ct (t \in \mathbb{R})$. These lines are called the *characteristics* of the equation.

$u_t + cu_x = 0$



$$x'(t) = \frac{dx(t)}{dt} = \frac{d(x_0+ct)}{dt} = c$$

$$f(t) = u(\overbrace{x_0+ct}, t)$$

$$f'(t) = \frac{\partial u}{\partial x}(x_0+ct, t)\cdot \underbrace{\frac{dx(t)}{dt}}_{=c} + \frac{\partial u}{\partial t}(x_0+ct,t)\cdot \underbrace{\frac{dt}{dt}}_{=1}$$

$$= c\frac{\partial u}{\partial x}(x_0+ct,t) + \frac{\partial u}{\partial t}(x_0+ct,t)$$

$= u_t + cu_x$ evaluated at (x_0+ct, t)

$= 0$ because $u_t + cu_x = 0$ everywhere.

Conclusion: if u satisfies $u_t + cu_x = 0$ then u is constant on the line $x = x_0 + ct$

The initial value problem

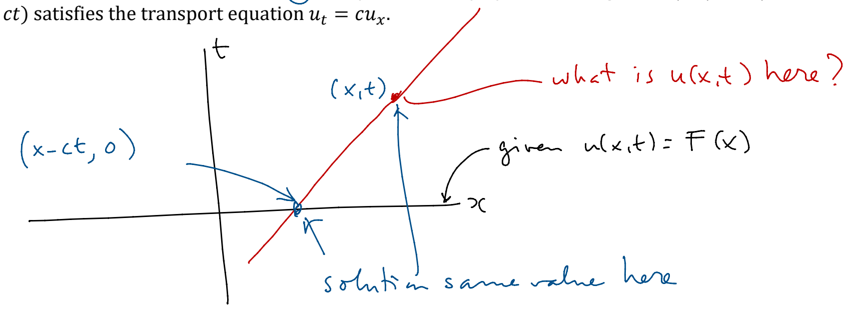
Suppose we are given the values $u(x,0) = F(x)$ of the solution at time $t = 0$ for all $x \in \mathbb{R}$.

Then for any $(x,t) \in \mathbb{R}^2$ one has $u(x,t) = u(x-ct,0) = F(x-ct)$.

In other words, *if there is a solution with the prescribed initial values then it must be $u(x,t) = F(x-ct)$.*

means: $F(x)$ and $F'(x)$ are continuous

On the other hand, if $F: \mathbb{R} \to \mathbb{R}$ is C^1 then you can verify by substituting that $u(x,t) = F(x - ct)$ satisfies the transport equation $u_t = cu_x$.



$$u(x,t) = u(x-ct, 0) = F(x-ct)$$

$u = F(x-ct)$ is a solution: $\quad u_x = \frac{\partial u}{\partial x} = \frac{\partial F(x-ct)}{\partial x} = F'(x-ct)\overbrace{\frac{\partial\, x-ct}{\partial x}}^{=1} = F'(x-ct)$

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial F(x-ct)}{\partial t} = F'(x-ct)\frac{\partial\, x-ct}{\partial t} = -cF'(x-ct)$$

$$\Rightarrow u_t + cu_x = -cF'(x-ct) + cF'(x-ct) = 0 \checkmark$$

1. There is a formula for the solution to $u_t + cu_x = 0$

   $u(x,t) = F(x-ct)$

2. The solution has an unknown function F

3. The formula is simple

ODE (from calculus)

Solve: $\frac{dx}{dt} = x \qquad (x = x(t))$

$\Leftrightarrow \frac{dx}{dt} - x = 0$

$\Leftrightarrow e^{-t}\frac{dx}{dt} - e^{-t}x = 0$

$\Leftrightarrow e^{-t}\frac{dx}{dt} + \frac{de^{-t}}{dt}x = 0$

$\Leftrightarrow \frac{d\, e^{-t}x}{dt} = 0$

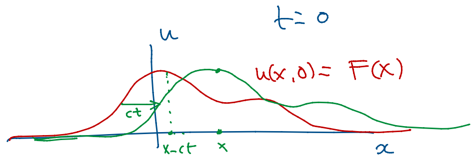
$\Leftrightarrow$ There is a number $C \in \mathbb{R}$ such that for all $t \in \mathbb{R}$: $e^{-t}x(t) = C$

$\Leftrightarrow x(t) = \underset{\substack{\uparrow \\ \text{unknown constant.}}}{C}\, e^t$

2. The solution has

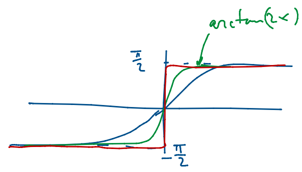
3. The formula is simple

$t = 0$



$u(x,t) = F(x-ct)$

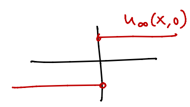
Solutions without derivatives



$u_n(x,0) = \arctan(nx)$

$u_\infty(x,0) \stackrel{\text{def}}{=} \lim_{n\to\infty} u_n(x,0) = \lim_{n\to\infty} \arctan(nx)$

$$= \begin{cases} +\frac{\pi}{2} & \text{when } x > 0 \\ -\frac{\pi}{2} & \text{when } x < 0 \end{cases}$$



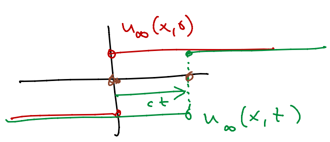
Solutions corresponding to $u_n(x,0)$:

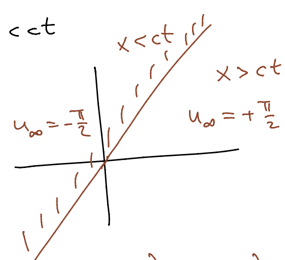
$u_n(x,t) = \arctan(n(x-ct))$ for all $n \in \mathbb{N}$, $x,t \in \mathbb{R}$

$u_\infty(x,t) = \lim_{n\to\infty} u_n(x,t) =$

$= \lim_{n\to\infty} \arctan[n(x-ct)]$

$$= \begin{cases} +\frac{\pi}{2} & \text{when } x > ct \\ -\frac{\pi}{2} & \text{when } x < ct \end{cases}$$





Question: is u_∞ a solution to $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$?

$u_\infty(x,t) = u_\infty(x-ct, 0)$