“Superficial” Problem Set

Problem 1

Let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the unit sphere. Consider the surface patch

$$\sigma(u,\theta) = \begin{pmatrix} \dfrac{\cos\theta}{\cosh u} \\ \dfrac{\sin\theta}{\cosh u} \\ \tanh u \end{pmatrix}, \qquad (u,\theta) \in S = \{(u,\theta) \in \mathbb{R}^2 \mid u \in \mathbb{R}, |\theta| < \pi\}$$

for the unit sphere.

(1) Compute the metric $(ds)^2 = E(u,\theta)(du)^2 + 2F(u,\theta)dud\theta + G(u,\theta)(d\theta)^2$.
(2) Show that the mapping from the strip S to the unit sphere Σ given by σ is conformal.
(3) Find a curve $\gamma : (0,\infty) \to \Sigma$ which starts at $A = \sigma(0,0)$, and makes a $45°$degree angle with every meridian it meets. [*Hint:* represent γ in the surface patch σ by setting $\gamma(t) = \sigma(t, \Theta(t))$.]

Problem 2

Consider a saddle surface S with surface patch $\sigma : \mathbb{R} \to \mathbb{R}^3$ given by

$$\sigma(u,v) = \begin{pmatrix} u \\ v \\ uv \end{pmatrix}.$$

(1) Compute the first fundamental form of σ.
(2) Find the area of the portion $\mathcal{R} = \{\sigma(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ of the surface parametrized by σ. (You may leave a double integral in the answer.)
(3) Compute the normal curvature κ_n and geodesic curvature κ_g of the curve γ on S given by $\gamma(t) = \sigma(t, at)$ where $a \in \mathbb{R}$ is a constant.
(4) Compute *second fundamental form*, the *principal curvatures*, and *principal curvature directions* of S at the point $(1, 1, 1)$.