Solutions to the first midterm were done in lecture following the midterm, except problem 2 which is solved here.

2(a). The curve is given by $\gamma(t)=\begin{pmatrix}\cos t\\ \sin t\\ \frac{1}{2}t^2\end{pmatrix}$. Hence

$$\gamma'(t)=\begin{pmatrix}-\sin t\\ \cos t\\ t\end{pmatrix},\quad \|\gamma'(t)\|=\sqrt{1+t^2},\quad \mathbf{T}(t)=\frac{1}{\sqrt{1+t^2}}\begin{pmatrix}-\sin t\\ \cos t\\ t\end{pmatrix}.$$

The binormal follows from

$$\mathbf{B}(t)=\frac{\gamma'\times\gamma''}{\|\gamma'\times\gamma''\|}.$$

Compute

$$\gamma''(t)=\begin{pmatrix}-\cos t\\ -\sin t\\ 1\end{pmatrix},\quad \gamma'\times\gamma''=\begin{vmatrix}\mathbf{i} & -\sin t & -\cos t\\ \mathbf{j} & \cos t & -\sin t\\ \mathbf{k} & t & 1\end{vmatrix}=\begin{pmatrix}\cos t+t\sin t\\ \sin t-t\cos t\\ -1\end{pmatrix}$$

whence $\|\gamma'\times\gamma''\|=\sqrt{2+t^2}$, so that

$$\mathbf{B}=\frac{1}{\sqrt{2+t^2}}\begin{pmatrix}\cos t+t\sin t\\ \sin t-t\cos t\\ -1\end{pmatrix}.$$

The unit normal follows from

$$\begin{aligned}\mathbf{N}&=\mathbf{B}\times\mathbf{T}\\ &=\frac{1}{\sqrt{(1+t^2)\,(2+t^2)}}\begin{vmatrix}\mathbf{i} & \cos t+t\sin t & -\sin t\\ \mathbf{j} & \sin t-t\cos t & \cos t\\ \mathbf{k} & -1 & t\end{vmatrix}\\ &=\frac{1}{\sqrt{(1+t^2)\,(2+t^2)}}\begin{pmatrix}t\sin t+(1-t^2)\cos t\\ -t\cos t+(1-t^2)\sin t\\ 1\end{pmatrix}\end{aligned}$$

2(b) The *Curvature* is given by

$$\kappa=\frac{\|\gamma'\times\gamma''\|}{\|\gamma'\|^3}=\frac{\sqrt{2+t^2}}{(1+t^2)^{3/2}}.$$

For the *torsion* one uses $\gamma'''=\begin{pmatrix}\sin t\\ -\cos t\\ 0\end{pmatrix}$ combined with the formula

$$\tau=\frac{(\gamma'\times\gamma'')\cdot\gamma'''}{\|\gamma'\times\gamma''\|^2}=\frac{\sin t(\cos t+t\sin t)-\cos t(\sin t-t\cos t)}{2+t^2}=\frac{t}{2+t^2}.$$

2(c). The tangent plane to the curve is the plane through$\gamma(0)$ parallel to $\mathbf{T}$ and $\mathbf{N}$, so it is perpendicular to $\mathbf{B}$. At $t=0$ we therefore find that $\mathbf{B}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$ is a normal to the tangent plane. Since $\gamma(0)=(1,0,0)$ must lie on this plane it has equation $x-z=1$.