LIOUVILLE'S FORMULA

for the geodesic curvature of a curve
in an orthogonal coordinate system.

Orthogonal coordinates. Let $\sigma : \mathcal{U}$ be a surface patch, and assume that the metric for σ is given by

$$(ds)^2 = E(u,v)(du)^2 + G(u,v)(dv)^2,$$

so that $F(u,v) = 0$, i.e. $\sigma_u \perp \sigma_v$.

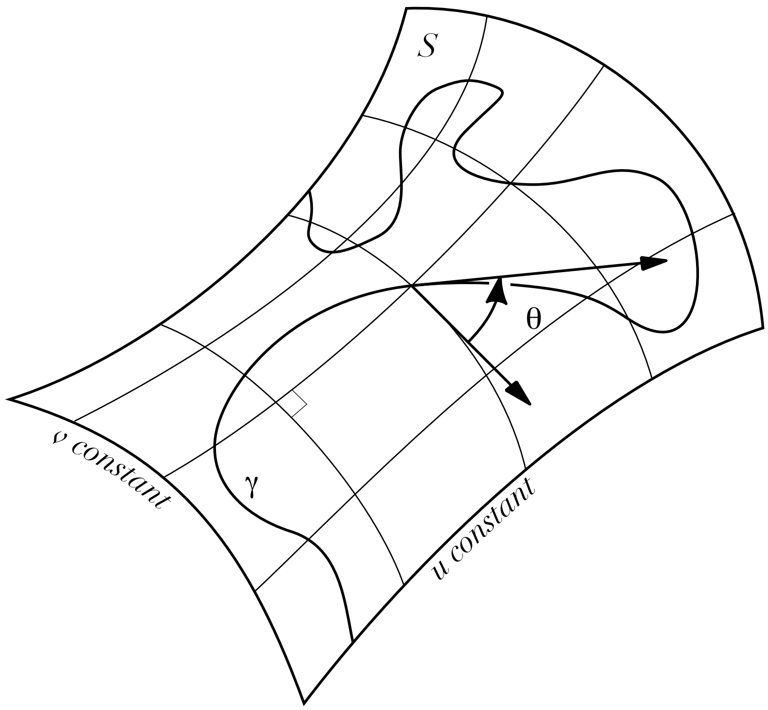

FIGURE 1. A curve in an orthogonal coordinate patch

Liouville's formula. If γ is a curve on the surface, and θ denotes the angle of intersection of γ with the curves $\{v = \text{constant}\}$, then its geodesic curvature is given by

$$\boxed{\kappa_g = \frac{d\theta}{ds} - \frac{(\sqrt{E})_v}{\sqrt{EG}} \cos\theta + + \frac{(\sqrt{G})_u}{\sqrt{EG}} \sin\theta}$$

This can be written as

$$\kappa_g = \frac{d\theta}{ds} + \kappa_u \cos\theta + \kappa_v \sin\theta$$

where

$$\kappa_u = -\frac{(\sqrt{E})_v}{\sqrt{EG}} \qquad \kappa_v = \frac{(\sqrt{G})_u}{\sqrt{EG}}$$

i.e. κ_u and κ_v are the geodesic curvatures of the curves $\{v = \text{const}\}$, and the curves $\{u = \text{const}\}$, respectively.

PROBLEMS

(1) Consider a surface of rotation,

$$\sigma(x,\theta) = x\,\mathbf{i} + R(x)\cos\theta\,\mathbf{j} + R(x)\sin\theta\,\mathbf{k}$$

(this is the surface obtained by rotating the graph of $y = R(x)$ around the x-axis.)

(a) Compute the metric of σ.

(b) Compute κ_x and κ_θ. (You can do this in two ways: (i) κ_x is the curvature of the curves $\{\theta = \text{const}\}$, so use Liouville's formula above; (ii) use the formula $\kappa_x = \mathbf{n} \cdot (\gamma'(t) \times \gamma''(t))/\|\gamma'(t)\|^3$, where $\gamma(t) = \sigma(t,\theta)$, with θ constant.)

(c) Consider the curve $\gamma(t) = \sigma(t,t)$. Draw the curve. Compute its geodesic curvature.

(2) Consider a surface $\sigma : \mathcal{U} \to \mathbb{R}^3$, where $\mathcal{U}$ is an open subset of the upper half plane, with metric

$$\mathbf{I} = \frac{(dx)^2 + (dy)^2}{y^2}.$$

Compute the geodesic curvature of the two curves from the second midterm, i.e. of the curve $\alpha(t) = \sigma(t,1)$, and of the curve $\beta(t) = \sigma\big(\sqrt{2}\cos t, \sqrt{2}\sin t\big)$.