GEODESIC CURVATURE PROBLEMS

Brief Solutions.

The following problems require you to review the definition of geodesic curvature of a curve γ on a surface. The most straightforward formula for κ_g in this context is

$$\kappa_g = \vec{\kappa} \cdot (\vec{n} \times \vec{T}) = \frac{\gamma''(t) \cdot (\vec{n} \times \gamma'(t))}{\|\gamma'(t)\|^3} \tag{1}$$

$\vec{n}$ being the surface normal.

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(1) Let γ be the "small circle" on the unit sphere

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

obtained by intersecting S with the plane $\{z = a\}$, where $a \in (-1, 1)$ is a constant.

(a) *Find a surface patch σ for S, and in this surface patch find a parametrization for γ.*
Spherical coordinates are given by

$$\sigma(\theta, \phi) = \begin{pmatrix} \cos\theta \sin\phi \\ \sin\theta \sin\phi \\ \cos\phi \end{pmatrix}$$

where $\theta \in (-\pi, \pi)$ and $0 < \phi < \pi$.
The small circle γ is given by $\theta(t) = t$, and $\phi(t) = \arccos a$, i.e.

$$\gamma(t) = \sigma(t, \arccos a) = \begin{pmatrix} \sqrt{1-a^2}\cos t \\ \sqrt{1-a^2}\sin t \\ a \end{pmatrix}$$

(b) *Compute the geodesic curvature κ_g at any point of the curve γ.*
Use formula (1) at the top of this page to get $\kappa_g = -\dfrac{a}{\sqrt{1-a^2}} = -\cot\phi$. *Details:* You could use $\vec{n} = c\sigma_\theta \times \sigma_\phi$ to compute the unit normal, but S is the sphere, so the normal is just $\vec{n} = \sigma(\theta, \phi)$. So you get

$$\gamma'(t) = \sqrt{1-a^2} \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \qquad \gamma''(t) = \sqrt{1-a^2} \begin{pmatrix} -\cos t \\ -\sin t \\ 0 \end{pmatrix}$$

$$\gamma'(t) \times \gamma''(t) = -(1-a^2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{n}(t, \arccos a) = \begin{pmatrix} \sqrt{1-a^2}\cos t \\ \sqrt{1-a^2}\sin t \\ a \end{pmatrix}$$

Finally this results in

$$\kappa_g = \frac{\vec{n} \cdot (\gamma' \times \gamma'')}{\|\gamma'\|^3} = \frac{-(1-a^2)a}{(1-a^2)^{3/2}} = -\frac{a}{\sqrt{1-a^2}}$$

Other possible solution: Try to argue directly from the definition of κ_g as the tangential component of the curvature vector $\vec{\kappa}$ of the *space curve* $\gamma \subset \mathbb{R}^3$. If you draw a picture, you see that the small circle has radius $\sqrt{1-a^2}$, so its curvature as a space curve is $\left(1-a^2\right)^{-1/2}$. Decompose this into normal and tangential parts, to get $\pm a/\sqrt{1-a^2}$ as geodesic curvature.

(c) *For which values of a does the curve γ have zero geodesic curvature?*
Only the equator (which is given by $a = 0$, or, equivalently $\phi = \pi/2$) has zero geodesic curvature.

(2) Let $\mathfrak{C}$ be the cylinder

$$\mathfrak{C} = \{(x, y, z) \mid x^2 + y^2 = 1\}$$

and let

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ at \end{pmatrix}$$

be a helix on $\mathfrak{C}$ ($a > 0$ is some constant.) Compute the geodesic curvature of γ.

Answer: The geodesic curvature of the helix is zero! Thus the helix is a geodesic on the cylinder. This is true for any value of the constant a. You can prove this by the same kind of calculation as in the previous problem, but you could also argue that (i) Geodesic curvature is an "intrinsic quantity," i.e. it is not change by an isometry, and (ii) you can map the cylinder isometrically onto the plane by

$$\sigma(u, v) = \begin{pmatrix} \cos v \\ \sin v \\ u \end{pmatrix}$$

and under this mapping the helix is mapped to a straight line. Hence its geodesic curvature must vanish.

(3) On the same cylinder $\mathfrak{C}$ as in the previous problem we consider the curve

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ h(t) \end{pmatrix},$$

where $h : \mathbb{R} \to \mathbb{R}$ is some smooth function.

(a) *Find the geodesic curvature of γ.*

One computes, and finds $\kappa_g = \dfrac{h''(t)}{\left(1 + h'(t)^2\right)^{3/2}}$. Alternatively, one could use the isometry between the cylinder and plane from problem 2(solution) to argue that the geodesic curvature of the curve γ on the cylinder must be the same as that of the graph of $u = h(v)$ in the plane.

(b) *Show that the geodesic curvature of γ vanishes if and only if $h(t) = at + b$ for certain constants a and b.*

In the expression $\kappa_g = h''(t)/\left(1 + h'(t)^2\right)^{3/2}$ from part (a) the denominator never vanishes. So $\kappa_g = 0$ holds if and only if $h''(t) = 0$. If the geodesic curvtaure of a curve vanishes everywhere on that curve, then one has $h''(t) = 0$ for all t, and hence $h(t) = at + b$.

(4) *Suppose a curve γ on a surface $S \subset \mathbb{R}^3$ has zero geodesic curvature, i.e. $\kappa_g = 0$. Must γ be a straight line?*

No, consider for example the equator on the sphere in problem 1. It has zero geodesic curvature (i.e. it's a geodesic), but it is clearly not a straight line.

(5) *Suppose a curve γ on a surface $S \subset \mathbb{R}^3$ has zero geodesic curvature, and zero normal curvature, (so $\kappa_g = \kappa_n = 0$ on the curve). Must γ be a straight line?*

If you consider γ as a space curve (curve in $\mathbb{R}^3$), then it has a curvature vector $\vec{\kappa}$ at each point. The normal and tangential companents to the surface S of this vector are κ_n and κ_g, respectively. If κ_n and κ_g *both* vanish, then the curvature vector $\vec{\kappa}$ of the space curve has to vanish, and therefore γ must be a straight line.

(6) *Suppose a surface $S \subset \mathbb{R}^3$ contains a straight line $\gamma \subset S$. Show that γ is a geodesic (i.e. a curve whose geodesic curvature vanishes.)*

This is the converse of the previous question. If γ is a straight line, then its curvature vector $\vec{\kappa}$ vanishes, and hence its tangential and normal components also vanish. So γ is a geodesic, which moreover satisfies the special property that its normal curvature vanishes.