GEODESIC CURVATURE PROBLEMS

The following problems require you to review the definition of geodesic curvature of a curve γ on a surface. The most straightforward formula for κ_g in this context is

$$\kappa_g = \vec{\kappa} \cdot (\vec{n} \times \vec{T}) = \frac{\gamma''(t) \cdot (\vec{n} \times \gamma'(t))}{\|\gamma'(t)\|^3}$$

$\vec{n}$ being the surface normal.

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(1) Let γ be the "small circle" on the unit sphere

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

obtained by intersecting S with the plane $\{z = a\}$, where $a \in (-1, 1)$ is a constant.
  (a) Find a surface patch σ for S, and in this surface patch find a parametrization for γ (suggestion: spherical coordinates, i.e. lattitude & longitude probably work best.)
  (b) Compute the geodesic curvature κ_g at any point of the curve γ.
  (c) For which values of a does the curve γ have zero geodesic curvature?

(2) Let $\mathfrak{C}$ be the cylinder

$$\mathfrak{C} = \{(x, y, z) \mid x^2 + y^2 = 1\}$$

and let

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ at \end{pmatrix}$$

be a helix on $\mathfrak{C}$ ($a > 0$ is some constant.)
Compute the geodesic curvature of γ.

(3) On the same cylinder $\mathfrak{C}$ as in the previous problem we consider the curve

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ h(t) \end{pmatrix},$$

where $h : \mathbb{R} \to \mathbb{R}$ is some smooth function.
  (a) Find the geodesic curvature of γ.
  (b) Show that the geodesic curvature of γ vanishes if and only if $h(t) = at + b$ for certain constants a and b.

(4) Suppose a curve γ on a surface $S \subset \mathbb{R}^3$ has zero geodesic curvature, i.e. $\kappa_g = 0$. Must γ be a straight line?

(5) Suppose a curve γ on a surface $S \subset \mathbb{R}^3$ has zero geodesic curvature, and zero normal curvature, (so $\kappa_g = \kappa_n = 0$ on the curve). Must γ be a straight line?

(6) Suppose a surface $S \subset \mathbb{R}^3$ contains a straight line $\gamma \subset S$. Show that γ is a geodesic (i.e. a curve whose geodesic curvature vanishes.)