ZERO CONTENT PROBLEMS

Definitions

Definition 0.1. *A **rectangle** is any set of the form* $\mathcal{R} = [a, b] \times [c, d]$.
Its area is $(b - a)(d - c)$.

Note that rectangles are, by definition, closed (they include their boundary points.)

Definition 0.2. *A bounded set* $S \subset \mathbb{R}^2$ *has **zero content** if for any* $\varepsilon > 0$ *one can find a finite number of rectangles* $\mathcal{R}_1, \ldots, \mathcal{R}_m \subset \mathbb{R}^2$ *such that*

$$S \subset \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_m$$

and

$$\textit{area}(\mathcal{R}_1) + \cdots + \textit{area}(\mathcal{R}_m) < \varepsilon.$$

Problems

1 Let $S, T \subset \mathbb{R}^2$ be two sets of content zero. Show that $S \cup T$ also has content zero.

2 Let $S \subset \mathbb{R}^2$ be a set of content zero. Show that every subset $T \subset S$ also has content zero.

3 Let $f : [a, b] \to \mathbb{R}$ be a nondecreasing function (i.e. if $x \leq x'$ then $f(x) \leq f(x')$). Show that the graph of f, i.e. the set

$$G = \big\{(x, f(x)) \mid a \leq x \leq b\big\}$$

has content zero.

Suggestion: For any (large) integer N consider the rectangles

$$\mathcal{R}_j = [x_{j-1}, x_j] \times [f(x_{j-1}), f(x_j)], \qquad (1 \leq j \leq N).$$

where

$$x_j = a + \frac{j}{N}(b - a)$$

There is a nice formula for the total area of the rectangles $\mathcal{R}_1, \ldots, \mathcal{R}_N$.

4 Show that the set $S = \{(x, y) \mid x^2 + 2y^2 = 3\}$ has content zero.