MATH 376
Concerning the 2nd midterm

The subjects of this midterm are:

- Line integrals (again). See the review sheet from the 1st midterm.
- The transformation formula for double integrals
- Surface integrals
- Stokes' Theorem

When studying for this exam you should do the problems from the homework, and those listed below. You should also study a few proofs (to be listed below).

Some sample problems

1 **[Transformation formula]** Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by $F(u, v) = (2uv, u^2 - v^2)$.

(***i***) Let $\mathcal{D}$ be the image of the rectangle $\mathcal{R} = \{(u, v) \mid a \leq u \leq b, a \leq v \leq b\}$. Draw $\mathcal{D}$.
(***ii***) Compute the area of $\mathcal{D}$.
(***iii***) Compute the volume of the solid obtained by rotating the region $\mathcal{D}$ around the y-axis.
(***iv***) Find the *moments of inertia* of $\mathcal{D}$. (Look the formulas up in the book!)

2 **[Integral in polar, spherical & cylindrical coordinates]** Let $\mathcal{S}$ be the spherical shell

$$\mathcal{S}_{a,b} = \{(x, y, z) \mid a^2 \leq x^2 + y^2 + z^2 \leq b^2\}.$$

(***i***) Compute the volume and moment of inertia

$$I = \iiint_{\mathcal{S}} \left(x^2 + y^2\right) \, \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

(***ii***) Compute the center of mass of the northern half of the shell, i.e. of

$$\mathcal{S}^+_{a,b} = \{(x, y, z) \in \mathcal{S}_{a,b} \mid z \geq 0\}.$$

3 **[Parametrization of a surface and tangent planes]** Consider the parametrization

$$\vec{\boldsymbol{r}}(u, v) = uv\vec{\boldsymbol{i}} + v^2\vec{\boldsymbol{j}} + (u^2 - v^2)\vec{\boldsymbol{k}}$$

(***i***) At which points in the (u, v)-plane is the parametrization $\vec{\boldsymbol{r}}$ regular (i.e. "smooth" in Apostol's terminology – look up the definition)?
(***ii***) Find the tangent plane at any smooth point $\vec{\boldsymbol{r}}(\bar{u}, \bar{v})$ on the surface.
(***iii***) At which point(s) $\vec{\boldsymbol{r}}(\bar{u}, \bar{v})$ is the tangent plane parallel to the plane with equation $2x + y + z = \frac{355}{113}$?
(***iv***) At which point(s) $\vec{\boldsymbol{r}}(\bar{u}, \bar{v})$ is the tangent plane parallel to the line through $(1, 2, 0)$ and $(0, 0, 1)$?
(***v***) At which point(s) $\vec{\boldsymbol{r}}(\bar{u}, \bar{v})$ does the tangent plane contain the origin?

4 **[Surface integrals]** Find the center of mass of the part of the sphere with radius $a > 0$ contained in the 1st octant, i.e. of

$$\mathcal{S} = \{(x, y, z) \mid x, y, z \geq 0, x^2 + y^2 + z^2 = a^2\}.$$

5 **[Surface integrals, Stokes Theorem]** Let S be the triangle with corners $A = (a, 0, 0)$, $B = (0, b, 0)$ and $C = (0, 0, c)$ (assume $a, b, c > 0$). We orient S so that its normal vector points upwards (has positive z component).
let T be the boundary of S.
(***i***) Find parametrizations for S and T.
(***ii***) Compute the area and center of mass of the triangle (see book for definitions.)
(***iii***) Compute $\iint_S \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{n}}\,\mathrm{d}S$ where $\vec{\boldsymbol{v}}\,(x, y, z) = \vec{\boldsymbol{i}} - 2\vec{\boldsymbol{j}}$
(***iv***) Compute $\oint_T \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{t}}\,\mathrm{d}s$ where $\vec{\boldsymbol{v}}\,(x, y, z) = y\vec{\boldsymbol{i}} - x\vec{\boldsymbol{j}} + z\vec{\boldsymbol{k}}$ in two different ways: first directly from the definitions, and again using Stokes' Theorem.

6 **[Surface area in PC]** Let the surface $\mathcal{S}$ be the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is given in polar coordinates by

$$z = F(r, \theta), \quad a \le r \le b, \quad \alpha \le \theta \le \beta$$

(i.e. $f(x, y) = F(r(x, y), \theta(x, y))$ where $r(x, y) = \sqrt{x^2 + y^2}$, $\tan\theta(x, y) = y/x$).
Derive a formula for the area of $\mathcal{S}$ in terms of an integral over the rectangle $[a, b] \times [\alpha, \beta]$ involving the function F and its partial derivatives F_r, F_θ.

7 **[More Stokes]** Consider the vectorfield

$$\vec{\boldsymbol{v}}\,(x, y, z) = \frac{y\vec{\boldsymbol{i}} - x\vec{\boldsymbol{j}}}{x^2 + y^2}.$$

(***i***) Compute $\mathbf{curl}\ \vec{\boldsymbol{v}}$.
(***ii***) Compute $\oint_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{t}}\,\mathrm{d}s$ where $\mathcal{C}$ is the unit circle in the xy plane, with counterclockwise orientation (seen from "above").
(***iii***) If $\mathcal{T}$ is the triangle with corners $P = (10, 0, 0)$, $Q = (-5, 10, 0)$ and $(5, 10, 0)$, then what is $\oint_{\mathcal{T}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{t}}\ \mathrm{d}s$?
(***iv***) If $\mathcal{T}$ is the triangle with corners $P = (10, 0, 0)$, $Q = (-5, 10, 1)$ and $(5, 10, -3)$, then what is $\oint_{\mathcal{T}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{t}}\ \mathrm{d}s$?

8 (***i***) *True or false:* If $\vec{\boldsymbol{v}}\,(x, y, z) = \nabla f(x, y, z)$ for some function with continuous second derivatives, then $\mathbf{curl}\ \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$.
(***ii***) *True or false:* If some continuously differentiable vector field $\vec{\boldsymbol{v}}\,(x, y, z)$ defined on a domain $\mathcal{D} \subset \mathbb{R}^3$ satisfies $\mathbf{curl}\ \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$, then there is a function $f : \mathcal{D} \to \mathbb{R}$ such that $\vec{\boldsymbol{v}}\,(x, y, z) = \nabla f(x, y, z)$.

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9 Prove Green's Theorem (in the plane, from the 1st midterm) assuming the domain is a rectangle $[a, b] \times [c, d]$.

10 State the transformation formula for double integrals.
Give a proof when the integrand is $f(x, y) \equiv 1$.