SUMMARY OF LINEAR TRANSFORMATIONS AND DETERMINANTS

1. Linear transformations

Definition of linear transformation; Nullspace and range of a linear transformation; Nullity plus Rank theorem. section

2. Theorems whose proof you should know

2.1. A linear transformation $T : V \to W$ is injective if and only if $\text{Null}(T) = \{0\}$.

2.2. If $T : V \to W$ has a *left inverse* $S : W \to V$, so that $ST = \text{Id}_V$, then T is injective.

2.3. If $T : V \to W$ has a *right inverse* $S : W \to V$, so that $TS = \text{Id}_W$, then T is surjective.
*Note that Apostol has a very unorthodox definition of left and right inverses, which we will **not** use. Instead, our definitions are those given above.*

2.4. If $\dim V = \dim W < \infty$, then for a linear map $T : V \to W$ one has

$$T \text{ is injective} \iff T \text{ is surjective} \iff T \text{ is bijective.}$$

2.5. If $T : V \to W$ is a linear map and $\dim V = \dim W < \infty$, then every left inverse of T is also a right inverse, and every right inverse is a left inverse.

3. Matrices

3.1. Given bases $\{e_i\}$ of V and $\{f_i\}$ of vector spaces V and W, you should be able to find the matrix M_T of a linear transformation $T : V \to W$.

3.2. Know that $M_{TS} = M_T M_S$.

3.3. Compute the inverse matrix M^{-1} of a given $n \times n$ matrix M using the Gauß-Jordan process.

4. Determinants.

This chapter contains a number of difficult proofs. You should concentrate on knowing the properties of the determinant, how to compute it, and its application to invertibility, independence and Cramer's rule.

4.1. Basic properties of $\det(A_1, \dots, A_n)$: Axioms 1, 2, 3 and Theorem 2.10 in Apostol's book.

4.2. $\det A^t = \det A$ and $\det(AB) = (\det A)(\det B)$.

4.3. A set of vectors $\{A_1, \dots, A_n\} \subset \mathbb{R}^n$ is linearly independent if and only if $\det(A_1, \dots, A_n) \neq 0$.

4.4. Know how to compute an $n \times n$ determinant by repeatedly adding rows to each other, or columns to each other, thereby reducing the determinant to upper triangular form.

4.5. Know how to compute an $n \times n$ determinant by expanding into minors along a column or row.

4.6. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if and only if $\det M_T \neq 0$, where M_T is the matrix of T with respect to the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$.

4.7. The formula for the inverse of a matrix, in terms of the cofactor matrix, and Cramer's rule.

5. Some Practice Problems

5.1. Let V be an n dimensional vector space and let $v_1, \dots, v_k$ be given vectors in V. Show that the map $T : \mathbb{R}^k \to V$ defined by

$$T(x_1, \dots, x_n) = x_1 v_1 + \cdots + x_k v_k$$

is surjective if and only if $\{v_1, \dots, v_k\}$ spans V. What is the dimension of the null space of T if $\{v_1, \dots, v_k\}$ spans V?

5.2. Let V be the vector space of all polynomials of degree 2 or less, i.e.

$$V = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}.$$

Find the matrix of the linear transformation $T : V \to V$ given by

$$(Tf)(x) \stackrel{\text{def}}{=} f(x - 1),$$

with respect to the basis

$$\varphi_1(x) = 1, \quad \varphi_2(x) = 1 + x, \quad \varphi_3(x) = x^2.$$

5.3. Compute the determinant $\det A$ where

$$A = \begin{pmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & -2 & 9 \end{pmatrix}.$$

Compute $\det(A^2)$.

5.4. You are given two matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -\mu & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5.4.1. Compute the matrix product $C = AB$.

5.4.2. Also compute $\det(C)$.

5.5. Find all values of x for which the matrix

$$M = \begin{pmatrix} x & x+1 \\ x-1 & 2 \end{pmatrix}$$

has an inverse, and then compute the inverse for those values of x.

5.6. For which values of α are the vectors

$$u = \begin{pmatrix} 1 \\ \alpha - 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ \alpha \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ \alpha \\ 0 \end{pmatrix}$$

linearly independent?

5.7. The following determinant defines a quadratic function of t

$$f(t) = \begin{vmatrix} 1 & t & 25 & 1 \\ t & 13 & 7 & -5 \\ \sqrt{2} & 5 & 2 & 3 \\ \frac{1}{2} & -7 & 4 & 2 \end{vmatrix}.$$

Compute the coefficient of t^2.