HIGHER ORDER DIFFERENTIAL EQUATIONS

1. Higher Order Equations

Consider the differential equation

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x)). \tag{1}$$

1.1. *The Existence and Uniqueness Theorem*

Suppose x_0 is a given "initial point" $x = x_0$, and suppose a_0, a_1, ..., a_{n-1} are given constants. Then there is exactly one solution to the differential equation (1) which satisfies the ***initial conditions***

$$y(x_0) = a_0, \quad y'(x_0) = a_1, \quad y''(x_0) = a_2, \quad \ldots, \quad y^{(n-1)}(x_0) = a_{n-1}. \tag{2}$$

Note that for an n^{th} order equation we can prescribe exactly n initial values. The proof of this theorem is difficult, and not part of math 320.

1.2. *The general solution*

If you try to solve the differential equation (1), and if everything goes well, then you will end up with a formula for the solution

$$y = y(x, c_1, c_2, \ldots, c_n)$$

which contains a number of constants. Often the way you got the solution leaves you with the possibility that there might be even more solutions than the ones you found. If you can show that there actually aren't any other solutions (i.e. your formula captures *all* solutions to the diffeq), then your solution is ***the general solution***.

One way to test if your solution is the general solution is to see if you can choose your constants $c_1, \ldots, c_n$ so that your solution satisfies the initial conditions (2). This means that you have to compute the derivatives $y, y', y'', \ldots, y^{(n-1)}$ of your solution, and then check if you can solve the equations

$$\begin{aligned} y(x_0, c_1, c_2, \ldots, c_n) &= a_0 \\ y'(x_0, c_1, c_2, \ldots, c_n) &= a_1 \\ y''(x_0, c_1, c_2, \ldots, c_n) &= a_2 \\ &\vdots \\ y^{(n-1)}(x_0, c_1, c_2, \ldots, c_n) &= a_{n-1} \end{aligned}$$

for $c_1, \ldots, c_n$. Note that you have n equations with n unknowns here. We'll do this for linear equations below.

2. Linear Equations

A differential equation of the form

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = f(x) \tag{3}$$

is called ***linear***. If the right hand side vanishes, i.e. if $f(x) = 0$, then the equation is called ***homogeneous***.

2.1. *The superposition principle*

Consider a linear homogeneous equation

$$y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_{n-1}(x)y'(x) + p_n(x)y(x) = 0. \tag{4}$$

The most important fact about linear homogeneous equations is ***the superposition principle***, which says:

- if $y_1(x)$ and $y_2(x)$ are solutions of (4), then so is $y_1 + y_2$.
- if $y_1(x)$ is a solution to (4), and if c is any constant, then $cy_1(x)$ is also a solution of (4).

Note the similarity with the definition of a linear subspace of $\mathbb{R}^n$!

The principle implies that if you have n solutions

$$y_1(x), y_2(x), \ldots, y_n(x)$$

then any linear combination

$$y_c(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \tag{5}$$

is also a solution, as long as the $c_1, \ldots, c_n$ are constants.

2.2. *The general solution and the Wronskian*

Once you have found n solutions $y_1, \ldots, y_n$ to the homogeneous equation (4), equation (5) gives you a formula for solutions to the differential equation which contains a number of constants. *Could this be the general solution?* To see if it is we check if we can use our formula to find the solution which satisfies the initial conditions (2). Compute the derivatives of our solution:

$$\begin{aligned} y_c(x) &= c_1 y_1(x) + \cdots + c_n y_n(x) \\ y_c'(x) &= c_1 y_1'(x) + \cdots + c_n y_n'(x) \\ &\vdots \\ y_c^{(n-1)}(x) &= c_1 y_1^{(n-1)}(x) + \cdots + c_n y_n^{(n-1)}(x) \end{aligned}$$

Thus to find a solution which satisfies the initial conditions

$$y(x_0) = a_0, \quad y'(x_0) = a_1, \quad y''(x_0) = a_2, \quad \ldots, \quad y^{(n-1)}(x_0) = a_{n-1}$$

we have to solve

$$\begin{aligned} c_1 y_1(x_0) + \cdots + c_n y_n(x_0) &= a_0 \\ c_1 y_1'(x_0) + \cdots + c_n y_n'(x_0) &= a_1 \\ &\vdots \\ c_1 y_1^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) &= a_{n-1} \end{aligned}$$

These are n linear equations for the n unknowns $c_1, \ldots, c_n$. In matrix form we can write the equations as

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ & & \vdots & \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}.$$

Our solution $y_c(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$ is the general solution if we can solve this system of equations for $c_1, \ldots, c_n$, no matter what initial data $a_0, \ldots, a_{n-1}$ we are given. We know that this happens exactly when the determinant of the matrix of coefficients is not zero. The conclusion is therefore:

If $y_1, \ldots, y_n$ are solutions to the homogeneous equation (4), then

$$y_c(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

is the general solution of that equation if and only if

$$W(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ & & \vdots & \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{vmatrix} \neq 0$$

The determinant $W(x_0)$ is called ***the Wronskian*** of the solutions $y_1, \ldots, y_n$.

2.3. *Which x_0 should you use?*

In the above theorem it doesn't matter which x_0 you choose. If $W(x_0) \neq 0$ for one choice of initial point x_0 then your solution $y_c(x)$ is the general solution, and $W(x_1) \neq 0$ for any other choice x_1. The Norwegian mathematician Nils Henrik Abel discovered a nice formula which relates the Wronskian $W(x)$ for different values of x. Abel's formula says

$$W(x_1) = W(x_0) e^{-\int_{x_0}^{x_1} p_1(x)dx},$$

and he found this by first showing that the Wronskian satisfies a first order differential equation

$$\frac{dW(x)}{dx} = -p_1(x) W(x),$$

known as Abel's differential equation.

3. **Two examples**

3.1. *Example*

For the fourth order differential equation

$$y^{(4)} - y = 0$$

a friend hands us four solutions, namely,

$$y_1(x) = e^x, \quad y_2(x) = e^{-x}, \quad y_3(x) = \sinh x, \quad y_4(x) = \cosh x.$$

We can check that they are solutions by substituting them in the diffeq. The superposition principle tells us that

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sinh x + c_4 \cosh x$$

is a solution for any choice of the constants $c_1, \ldots, c_4$. *Is this the general solution?* To answer this question we compute the Wronskian

$$W(x) = \begin{vmatrix} e^x & e^{-x} & \sinh x & \cosh x \\ (e^x)' & (e^{-x})' & \sinh' x & \cosh' x \\ (e^x)'' & (e^{-x})'' & \sinh'' x & \cosh'' x \\ (e^x)''' & (e^{-x})''' & \sinh''' x & \cosh''' x \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} & \sinh x & \cosh x \\ e^x & -e^{-x} & \cosh x & \sinh x \\ e^x & e^{-x} & \sinh x & \cosh x \\ e^x & -e^{-x} & \cosh x & \sinh x \end{vmatrix} \quad \text{(remember: } \sinh' = \cosh, \cosh' = \sinh)$$

The first and third rows in this determinant are equal, so the conclusion is

$$W(x) = 0.$$

Our solution is ***not*** the general solution!!

3.2. *Example, continued*

Another friend gave us two more solutions to

$$y^{(4)}(x) - y(x) = 0,$$

namely, $y_5(x) = \sin x$ and $y_6(x) = \cos x$. Therefore we now know that

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$$

is a solution. Could this be the general solution? Once again, we compute the Wronskian:

$$W(x) = \begin{vmatrix} e^x & e^{-x} & \sin x & \cos x \\ (e^x)' & (e^{-x})' & \sin' x & \cos' x \\ (e^x)'' & (e^{-x})'' & \sin'' x & \cos'' x \\ (e^x)''' & (e^{-x})''' & \sin''' x & \cos''' x \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} & \sin x & \cos x \\ e^x & -e^{-x} & \cos x & -\sin x \\ e^x & e^{-x} & -\sin x & -\cos x \\ e^x & -e^{-x} & -\cos x & \sin x \end{vmatrix} \quad \text{(remember: } \sin' = \cos, \cos' = -\sin)$$

Factor out the e^x and e^{-x} from this determinant:

$$\begin{aligned}
W(x) &= e^x \cdot e^{-x} \cdot \begin{vmatrix} 1 & 1 & \sin x & \cos x \\ 1 & -1 & \cos x & -\sin x \\ 1 & 1 & -\sin x & -\cos x \\ 1 & -1 & -\cos x & \sin x \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 & \sin x & \cos x \\ 1 & -1 & \cos x & -\sin x \\ 2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{vmatrix} && \begin{bmatrix} \text{add Row 1 to Row 3} \\ \text{and Row 2 to Row 4} \end{bmatrix} \\
&= \begin{vmatrix} 1 & 1 & \sin x & \cos x \\ 1 & -1 & \cos x & -\sin x \\ 2 & 2 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix} && [\text{add Row 3 to Row 4}] \\
&= (-4)(+2) \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} \\
&= 8.
\end{aligned}$$

This time the Wronskian is not zero, so

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x$$

is the general solution.

4. **Answers to some problems in the book**

§5.2 – Problem 13

We write $\cdots$ for entries in the determinant we don't need to know.

$$W = \begin{vmatrix} e^x & e^{-x} & e^{-2x} \\ e^x & -e^{-x} & -2e^{-2x} \\ e^x & e^{-x} & 4e^{-2x} \end{vmatrix} = e^{x-x-2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = e^{-2x} \begin{vmatrix} 1 & \cdots & \cdots \\ 0 & -2 & \cdots \\ 0 & 0 & 3 \end{vmatrix} = -6e^{-2x}$$

§5.2 – Problem 15

$$\begin{aligned}
W &= \begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & (x+1)e^x & (x^2+2x)e^x \\ e^x & (x+2)e^x & (x^2+4x+2)e^x \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 1 & x+1 & x^2+2x \\ 1 & x+2 & x^2+4x+2 \end{vmatrix} = \\
&= e^{3x} \begin{vmatrix} 1 & \cdots & \cdots \\ 0 & 1 & 2x \\ 0 & 2 & 4x+2 \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & \cdots & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & 2 \end{vmatrix} \\
&= 2e^{3x}.
\end{aligned}$$

§5.2 – Problem 19

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \qquad \text{(factor out an } x \text{ from first row and last column)}$$

$$= x \cdot x \begin{vmatrix} 1 & x & x \\ 1 & 2x & 3x \\ 0 & 2 & 6 \end{vmatrix} = x^2 \begin{vmatrix} 1 & \cdots & \cdots \\ 0 & x & 2x \\ 0 & 2 & 6 \end{vmatrix} = x^2 \begin{vmatrix} x & 2x \\ 2 & 6 \end{vmatrix} = x^2(6x - 4x) = 2x^3.$$

§5.2 – Problem 25

Given: y_1 satisfies $Ly_1 = f(x)$ and y_2 satisfies $Ly_2 = g(x)$, where $Ly = y'' + py' + qy$, by definition.

To be shown: $L(y_1 + y_2) = f(x) + g(x)$.

Proof: By definition we have

$$L(y_1 + y_2) = (y_1 + y_2)'' + p(y_1 + y_2)' + q(y_1 + y_2).$$

Since $(y_1 + y_2)' = y_1' + y_2'$, and $(y_1 + y_2)'' = y_1'' + y_2''$ we get

$$L(y_1 + y_2) = y_1'' + y_2'' + py_1' + py_2' + qy_1 + qy_2.$$

Rearrange the terms and we get

$$L(y_1 + y_2) = y_1'' + py_1' + qy_1 + y_2'' + py_2' + qy_2 = Ly_1 + Ly_2.$$

This is what we had to show.

§5.2 – Problem 31(a)

If y is a solution of $y'' + py + qy = 0$, and if you are given the values of y and y' at $x = a$, then the differential equation tells you that

$$y''(a) = -py'(a) - qy(a).$$

If p and q are not constant (the book is vague about this), then one should write

$$y''(a) = -p(a)y'(a) - q(a)y(a).$$

§5.2 – Problem 31(b)

The problem: *Prove the equation $y'' - 2y' - 5y = 0$ has a solution satisfying the conditions $y(0) = 1, y'(0) = 0, y''(0) = C$* ***if and only if*** *$C = 5$.*

The "if and only if" means that you have to do two things:

(1) assuming there is a solution show that $C = 5$,
(2) assuming $C = 5$, show there is a solution.

The first part (1) here is a special case of problem (31a). If there is a solution, then the differential equation implies that at $x = 0$ one has

$$y''(0) = 2y'(0) + 5y(0) \implies C = 2 \times 0 + 5 \times 1 = 5.$$

For the second part (2) you have two options.

First approach: you could calculate the solution to

$$(\dagger) \qquad y'' - 2y' - 5y = 0, \qquad y(0) = 1, y'(0) = 0.$$

That shows that a solutions exists (you've found it). The solution you have must then satisfy $y''(0) = 5 = C$.

The other approach is to apply the existence theorem: Theorem 2, page 302 says that (†) has a solution. The differential equation then implies that this solution must satisfy $y''(0) = C$, because $C = 5$. In this approach you don't have to calculate the solution.