PRACTICE PROBLEMS FOR THE MATH 320 FINAL, SPRING MMXI

Instructions: For every problem give a short outline of your approach before you compute anything. Then solve the problem according to your plan.

1. (***i***) For which values of the constant α does the system

$$\begin{aligned} x + 3y - 4z + w &= 1 \\ 2x - 3y - 4z + w &= \alpha \\ -x + 6y &= 1 \end{aligned}$$

have a solution? Find the general solution when there is one.

(***ii***) This problem looks like the previous one, but the answer is different: Use row reduction to decide for which values of α the following system of equations has 0, 1, or ∞ many solutions: NEW!

$$\begin{aligned} x + 3y - 4z + w &= 1 \\ 2x - 3y - 4z + w &= \alpha \\ -x + 6y + \alpha z &= 1 \end{aligned}$$

(You do not have to compute the solutions – they are quite ugly).

2. (***i***) State the definition of a *linear subspace* of $\mathbb{R}^n$.

(***ii***) State the definition of *an eigenvector* of a matrix A.

(***iii***) Let S be the set of solutions $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ to $2x_1 + 3x_2 = 1$ i.e., define

$$S = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^2 \mid 2x_1 + 3x_2 = 1 \right\}.$$

Prove or disprove: S is a linear subspace of $\mathbb{R}^2$.

3. For which values of the constant α are the vectors

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} \alpha \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 2\alpha \end{bmatrix},$$

linearly dependent?

4. Find the solution to

$$\boldsymbol{x}''(t) + A\,\boldsymbol{x} = 0, \text{ where } A = \begin{bmatrix} 7 & -4 \\ -4 & 13 \end{bmatrix}$$

with initial data

$$\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}'(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

5. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 3 & 2 \\ -2 & -3 & 0 & 4 \end{bmatrix}$$

6. For which values of α does the matrix

$$A = \begin{bmatrix} 2 & \alpha \\ 1 & 3+\alpha \end{bmatrix}$$

have only real eigenvalues?

7. Find the eigenvalues and eigenvectors of the matrices

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 3 & -6 \\ 4 & 2 & -4 \\ 4 & 3 & -5 \end{bmatrix}$$

(or any of the other matrices in problems 1…32 of §6.1, p.374-375)

8. The eigenvalues and vectors of the matrix A are

$$\lambda_1 = -4,\ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } \lambda_2 = -6,\ \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(***i***) If $\boldsymbol{a} = 12\boldsymbol{v}_1 + 24\boldsymbol{v}_2$ then compute the first component of $A^2\boldsymbol{a}$.
(***ii***) Compute $A^{-1}\boldsymbol{a}$ (same $\boldsymbol{a}$ as above).
(***iii***) Use the method of undetermined parameters to solve

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) + \begin{bmatrix} 120\sin 5t \\ 0 \end{bmatrix}, \text{ with } \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(***iv***) Use the method of undetermined parameters to solve

$$\boldsymbol{x}''(t) = A\boldsymbol{x}(t) + \begin{bmatrix} 120\sin 5t \\ 0 \end{bmatrix}, \text{ with } \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{x}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

9. Find the general solution to the inhomogeneous system of linear differential equations

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{i\omega t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ where } A = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \text{ and } \omega \text{ is arbitrary.}$$

Your solution should be valid for ***almost*** all possible values of the forcing frequency ω: if there are one or two values for which your solution is not valid, indicate those values.

ANSWERS

(1i) Plan: we row reduce the system of equations.
Here is the computation of the row reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 1 \\ 2 & -3 & -4 & 1 & \alpha \\ -1 & 6 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 1 \\ 0 & -9 & 4 & -1 & \alpha-2 \\ 0 & 9 & -4 & 1 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 1 \\ 0 & -9 & 4 & -1 & \alpha-2 \\ 0 & 0 & 0 & 0 & \alpha \end{array}\right]$$

The third equation shows that there are no solutions when $\alpha \neq 0$.
When $\alpha = 0$ you get this system

$$\left[\begin{array}{cccc|c} 1 & 3 & -4 & 1 & 1 \\ 0 & -9 & 4 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{8}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{4}{9} & \frac{1}{9} & \frac{2}{9} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution has two parameters. If we call them s and t and choose $z = s$ and $w = t$, then the solution is

$$x = \frac{1}{3} + \frac{8}{3}s - \frac{2}{3}t, \quad y = \frac{2}{9} + \frac{4}{9}s - \frac{1}{9}t, \quad z = s, \quad w = t.$$

(2i) A set S of vectors in $\mathbb{R}^n$ is called a subspace if

- For any pair of vectors $\boldsymbol{x} \in S, \boldsymbol{y} \in S$ one has $\boldsymbol{x} + \boldsymbol{y} \in S$, and
- For any vector $\boldsymbol{x} \in S$ and any number t one has $t\boldsymbol{x} \in S$.

(2ii) If A is a matrix then an eigenvector of A is a vector $\boldsymbol{v}$ which satisfies these conditions:

- $\boldsymbol{v} \neq \boldsymbol{0}$
- $A\boldsymbol{v} = \lambda\boldsymbol{v}$ for some number λ.

Note: don't forget the first condition $\boldsymbol{v} \neq \boldsymbol{0}$, it is just as important as the equation $A\boldsymbol{v} = \lambda\boldsymbol{v}$.

(2iii) In this case you could draw the subspace S: it's a line in the plane, and it doesn't go through the origin. To actually show that S is not a subspace you find two vectors $\boldsymbol{x}, \boldsymbol{y} \in S$ whose sum does not belong to S, or you find a vector $\boldsymbol{x} \in S$ and a number t such that $t\boldsymbol{x}$ does not belong to S. You only have to find one such example to show that S is not a linear subspace. Possible answers in this problem are:

- $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$ both belong to S, but $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$ does not belong to S, therefore S is not a linear subspace.
- The vector $\boldsymbol{x} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ belongs to S, but the vector $2\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not, therefore S is not a linear subspace.
- The set S does not contain the zero vector because $(x_1, x_2) = (0, 0)$ does not satisfy the equation $2x_1 + 3x_2 = 1$. If S were a linear subspace then it would have to contain the zero vector (this was proved in class). Therefore S is not a linear subspace.

(3) You can use the definition of linear independence, or you can use the fact that we have 3 vectors in $\mathbb{R}^3$: to test if n vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ in $\mathbb{R}^n$ are linearly independent, you compute the determinant you get by putting the column vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ together in an $n \times n$ matrix. If the determinant is zero the vectors are dependent, otherwise they are independent.

By doing this you will get the answer $\alpha = \pm\frac{1}{2}$.

(4) First of all note that the problem is written in a slightly different form from the standard formulation in the book, where the solution of $\boldsymbol{x}'' = A\boldsymbol{x}$ is presented. In this problem we are dealing with $\boldsymbol{x}'' = -A\boldsymbol{x}$. To remember the formula for the solution, let's go through the reasons behind them: The general solution to the homogeneous equation is a linear combination of solutions of the form

$$\boldsymbol{x}(t) = \sin(\alpha t)\boldsymbol{v} \text{ and } \boldsymbol{x}(t) = \cos(\alpha t)\boldsymbol{v}.$$

This $\boldsymbol{x}(t)$ is a solution of $\boldsymbol{x}'' = -A\boldsymbol{x}$ if

$$\begin{aligned} & \boldsymbol{x}'' = -A\boldsymbol{x} \\ \Longrightarrow \quad & -\alpha^2 \sin(\alpha t)\boldsymbol{v} = -\sin(\alpha t)A\boldsymbol{v} \\ \Longrightarrow \quad & \alpha^2\boldsymbol{v} = A\boldsymbol{v}. \end{aligned}$$

So $\boldsymbol{v}$ has to be an eigenvector of A and α^2 must be the corresponding eigenvalue.

Computation of the eigenvectors/values of A:

$$\begin{vmatrix} 7-\lambda & -4 \\ -4 & 13-\lambda \end{vmatrix} = \lambda^2 - 20\lambda + 75 = (\lambda-5)(\lambda-15).$$

The eigenvalues are 5 and 15. Computing the corresponding eigenvectors leads to

$$\lambda_1 = 5, \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \lambda_2 = 15, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The general solution to the homogeneous equation. The matrix A has two eigenvalues and therefore there are two choices of α, namely $\alpha = \sqrt{5}$ and $\alpha = \sqrt{15}$. For each α we get two basic solution, one with $\cos \alpha t \boldsymbol{v}$ and one with $\sin \alpha t \boldsymbol{v}$. The general solution to the homogeneous equation is a linear combination of these:

$$\begin{aligned} \boldsymbol{x}_h(t) &= P\cos(\sqrt{5}t)\boldsymbol{v}_1 + Q\sin(\sqrt{5}t)\boldsymbol{v}_1 + R\cos(\sqrt{15}t)\boldsymbol{v}_2 + S\sin(\sqrt{15}t)\boldsymbol{v}_2 \\ &= \big(P\cos(\sqrt{5}t) + Q\sin(\sqrt{5}t)\big)\boldsymbol{v}_1 + \big(R\cos(\sqrt{15}t) + S\sin(\sqrt{15}t)\big)\boldsymbol{v}_2. \end{aligned}$$

Matching the initial conditions. We have

$$\boldsymbol{x}_h(0) = P\boldsymbol{v}_1 + R\boldsymbol{v}_2.$$

Since the prescribed initial value is $\boldsymbol{x}(0) = \mathbf{0}$ we find that $P\boldsymbol{v}_1 + R\boldsymbol{v}_2 = \mathbf{0}$, so that

$$P = R = 0.$$

To match the intiial velocity $\boldsymbol{x}'(0) = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, we compute

$$\boldsymbol{x}_h'(t) = \big(-\sqrt{5}P\sin(\sqrt{5}t) + \sqrt{5}Q\cos(\sqrt{5}t)\big)\boldsymbol{v}_1 + \big(-\sqrt{15}R\sin(\sqrt{15}t) + \sqrt{15}S\cos(\sqrt{15}t)\big)\boldsymbol{v}_2$$

and set $t = 0$,

$$\boldsymbol{x}_h'(0) = \sqrt{5}Q\boldsymbol{v}_1 + \sqrt{15}S\boldsymbol{v}_2 \stackrel{?}{=} \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right].$$

To be able to compare both sides we write $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ as a combination of the eigenvectors: if $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] = f_1\boldsymbol{v}_1 + f_2\boldsymbol{v}_2$ then f_1, f_2 are found by solving

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -2 & 2 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 4/5 \\ 0 & 1 & -3/5 \end{array}\right] \implies f_1 = \tfrac{4}{5}, f_2 = -\tfrac{3}{5}.$$

So we have to solve

$$\boldsymbol{x}_h'(0) = \sqrt{5}Q\boldsymbol{v}_1 + \sqrt{15}S\boldsymbol{v}_2 \stackrel{?}{=} \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] = \tfrac{4}{5}\boldsymbol{v}_1 - \tfrac{3}{5}\boldsymbol{v}_2.$$

We get

$$Q = \frac{4}{5\sqrt{5}}, \qquad S = -\frac{3}{5\sqrt{15}}.$$

The solution is

$$\boldsymbol{x}(t) = \frac{4}{5\sqrt{5}}\sin(\sqrt{5}t)\boldsymbol{v}_1 - \frac{3}{5\sqrt{15}}\sin(\sqrt{15}t)\boldsymbol{v}_2.$$

(5) They are $1, 2, 3, 4$.

(6) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & \alpha \\ 1 & 3+\alpha-\lambda \end{vmatrix} = \lambda^2 - (5+\alpha)\lambda + 6 + \alpha.$$

Therefore the eigenvalues of A are

$$\lambda_{1,2} = \frac{5+\alpha \pm \sqrt{D}}{2}$$

where

$D = (5+\alpha)^2 - 4(6+\alpha) = 25 + 10\alpha + \alpha^2 - 24 - 4\alpha = 1 + 6\alpha + \alpha^2 = (\alpha+3)^2 - 8.$

The eigenvalues of A are real if the discriminant $D \geq 0$, i.e. if either $\alpha + 3 \geq 2\sqrt{2}$ or $\alpha + 3 \leq -2\sqrt{2}$. So A has real eigenvalues if

$$\alpha \geq -3 + 2\sqrt{2} \text{ or } \alpha \leq -3 - 2\sqrt{2}.$$

(7) They are

$$\lambda_1 = -1, \boldsymbol{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda_2 = 6, \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda_3 = 2, \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(8i) Computation of $A\boldsymbol{a}$:

$$\begin{aligned} A\boldsymbol{a} &= A(12\boldsymbol{v}_1 + 24\boldsymbol{v}_2) \\ &= 12A\boldsymbol{v}_1 + 24A\boldsymbol{v}_2 \\ &= 12(-4)\boldsymbol{v}_1 + 24(-6)\boldsymbol{v}_2 \\ &= -48\boldsymbol{v}_1 - 144\boldsymbol{v}_2 \\ &= -48\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 144\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -48 - 2\times 144 \\ -48\times 2 - 144 \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \end{bmatrix} \end{aligned}$$

To compute $A^2\boldsymbol{a}$ repeat this procedure. You get:

$$A^2\boldsymbol{a} = 12(-4)^2\boldsymbol{v}_1 + 24(-6)^2\boldsymbol{v}_2 = \text{etc}\dots$$

The problem asks for *the first component* of $A^2\boldsymbol{a}$. By definition this is the top entry you get when you write $A^2\boldsymbol{a}$ as a column vector. Thus

$$1^{\text{st}} \text{ component of } A^2\boldsymbol{a} = 12(-4)^2 \times 1 + 24(-6)^2 \times 2 = \text{etc}\dots$$

since the first component of $\boldsymbol{v}_1 = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ is 1, and the first component of $\boldsymbol{v}_2 = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ happens to be 2.

(8ii) Let $\boldsymbol{y} = A^{-1}\boldsymbol{a}$: this means that $\boldsymbol{y}$ is the solution to $A\boldsymbol{y} = \boldsymbol{a}$. Write $\boldsymbol{y}$ in terms of the eigenvectors, $\boldsymbol{y} = y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2$. Then we have to solve

$$A\boldsymbol{y} = -4y_1\boldsymbol{v}_1 - 6y_2\boldsymbol{v}_2 \stackrel{?}{=} \boldsymbol{a} = 12\boldsymbol{v}_1 + 24\boldsymbol{v}_2.$$

This implies $-4y_1 = 12$ and $-6y_2 = 24$. Solve this to get $y_1 = -3$, $y_2 = -4$, and hence

$$\boldsymbol{y} = -3\boldsymbol{v}_1 - 4\boldsymbol{v}_2 = \begin{bmatrix} -11 \\ -10 \end{bmatrix}.$$

(8iii) The general solution to the homogeneous first order equation is

$$\boldsymbol{x}_h(t) = c_1 e^{-4t}\boldsymbol{v}_1 + c_2 e^{-6t}\boldsymbol{v}_2.$$

To find a particular solution we note that the RHS has the form $\sin(5t)\boldsymbol{a}$ for some constant vector. Trying a similar expression leads to cosines when you compute $\boldsymbol{x}'$, so we must include terms with $\cos 5t$. Our first guess would be

$$\boldsymbol{x}_p = \cos 5t\ \boldsymbol{m} + \sin 5t\ \boldsymbol{n},$$

for undetermined constant vectors $\boldsymbol{m}, \boldsymbol{n}$. Since we will have to multiply this vector with A, it is better to write it as a combination of eigenvectors. So we try

$$\boldsymbol{x}_p = \big(P\cos 5t + Q\sin 5t\big)\boldsymbol{v}_1 + \big(R\cos 5t + S\sin 5t\big)\boldsymbol{v}_2.$$

Substitute in the equation:

$$\begin{aligned}
\boldsymbol{x}_p' &= \big(5Q\cos 5t - 5P\sin 5t\big)\boldsymbol{v}_1 + \big(5S\cos 5t - 5R\sin 5t\big)\boldsymbol{v}_2\\
A\boldsymbol{x}_p &= \big(-4P\cos 5t - 4Q\sin 5t\big)\boldsymbol{v}_1 + \big(-6R\cos 5t - 6S\sin 5t\big)\boldsymbol{v}_2\\
\boldsymbol{x}_p' - A\boldsymbol{x}_p &= \big((4P+5Q)\cos 5t + (-5P+4Q)\sin 5t\big)\boldsymbol{v}_1\\
&\quad + \big((6R+5S)\cos 5t + (-5R+6S)\sin 5t\big)\boldsymbol{v}_2 \qquad (1)\\
&\stackrel{?}{=} \sin 5t \begin{bmatrix}120\\0\end{bmatrix}.
\end{aligned}$$

Write $\left[\begin{smallmatrix}120\\0\end{smallmatrix}\right]$ as a combination of the eigenvectors:

$$\begin{bmatrix}120\\0\end{bmatrix} = f_1\boldsymbol{v}_1 + f_2\boldsymbol{v}_2 \implies f_1 = -40, f_2 = 80, \text{ so } \begin{bmatrix}120\\0\end{bmatrix} = -40\boldsymbol{v}_1 + 80\boldsymbol{v}_2.$$

The RHS of the equation is therefore

$$\boldsymbol{x}' - A\boldsymbol{x} = -40\sin 5t\boldsymbol{v}_1 + 80\sin 5t\boldsymbol{v}_2.$$

Compare with (1) and you get these equations for P, Q, R, S:

$$4P + 5Q = 0, \quad -5P + 4Q = -40, \quad 6R + 5S = 0, \quad -5R + 6S = 80.$$

The solution is:

$$P = \frac{200}{41}, \quad Q = -\frac{160}{41}, \quad R = -\frac{400}{61}, \quad S = \frac{480}{61}.$$

Substituting these (ugly, un-examlike) numbers in the formula for $\boldsymbol{x}_p$ gives you the particular solution.

The general solution to the inhomogeneous equation is

$$\begin{aligned}
\boldsymbol{x}_g(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)\\
&= \Big(c_1 e^{-4t} + P\cos 5t + Q\sin 5t\Big)\boldsymbol{v}_1 + \Big(c_2 e^{-6t} + R\cos 5t + S\sin 5t\Big)\boldsymbol{v}_2.
\end{aligned}$$

Its initial value is

$$\boldsymbol{x}_g(0) = (c_1 + P)\boldsymbol{v}_1 + (c_2 + R)\boldsymbol{v}_2.$$

If we want the solution with initial value $\boldsymbol{x}(0) = \mathbf{0}$, then we have to choose

$$c_1 = -P, \qquad c_2 = -R,$$

with P, R as above.

(8iv) The homogeneous equation.

$$\boldsymbol{x}_h(t) = \big(P\cos 2t + Q\sin 2t\big)\boldsymbol{v}_1 + \big(R\cos\sqrt{6}t + S\sin\sqrt{6}t\big)\boldsymbol{v}_2.$$

A particular solution. Try $\boldsymbol{x}_p = \sin 5t\boldsymbol{a}$. There is no need to include cosine since we only have second derivatives in the problem. But is better to write $\boldsymbol{a}$ in terms of the eigenvectors. So we try

$$\boldsymbol{x}_p = K\sin 5t\boldsymbol{v}_1 + L\sin 5t\boldsymbol{v}_2.$$

Substitute in $\boldsymbol{x}'' - A\boldsymbol{x}$ and you get

$$\begin{aligned}\boldsymbol{x}_p'' - A\boldsymbol{x}_p &= -25K\sin 5t\boldsymbol{v}_1 - 25L\sin 5t\boldsymbol{v}_2 - \big(-4K\sin 5t\boldsymbol{v}_1 - 6L\sin 5t\boldsymbol{v}_2\big)\\ &= -21K\sin 5t\boldsymbol{v}_1 - 19L\sin 5t\boldsymbol{v}_2.\end{aligned}$$

This should equal $\sin 5t\left[\begin{smallmatrix}120\\0\end{smallmatrix}\right]$. To compare write that vector in terms of the eigenvectors:

$$\begin{bmatrix}120\\0\end{bmatrix} = -40\boldsymbol{v}_1 + 80\boldsymbol{v}_2.$$

(Same as in part (iii) of this problem.)

Thus K, L satisfy

$$-21K = -40 \text{ and } -19L = 80, \implies K = \frac{40}{21}, \quad L = -\frac{80}{19}.$$

Therefore our particular solution is

$$\boldsymbol{x}_p = \frac{40}{21}\sin 5t\boldsymbol{v}_1 - \frac{80}{19}\sin 5t\boldsymbol{v}_2.$$

The solution with the right initial data. The general solution is

$$\begin{aligned}\boldsymbol{x}_g(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)\\ &= \Big(P\cos 2t + Q\sin 2t + \frac{40}{21}\sin 5t\Big)\boldsymbol{v}_1 + \Big(R\cos\sqrt{6}t + S\sin\sqrt{6}t - \frac{80}{19}\sin 5t\Big)\boldsymbol{v}_2.\end{aligned}$$

We still have to find P, Q, R, S.

Set $t = 0$ in $\boldsymbol{x}_g(t)$:

$$\boldsymbol{x}_g(0) = P\boldsymbol{v}_1 + R\boldsymbol{v}_2.$$

The initial condition $\boldsymbol{x}(0) = \mathbf{0}$ implies

$$P = R = 0.$$

We also have

$$\boldsymbol{x}_g'(0) = \Big(2Q + 5\times\frac{40}{21}\Big)\boldsymbol{v}_1 + \Big(\sqrt{6}S - 5\times\frac{80}{19}\Big)\boldsymbol{v}_2.$$

The initial condition $\boldsymbol{x}'(0) = \mathbf{0}$ implies that

$$Q = -\frac{5}{2}\times\frac{40}{21}, \text{ and } S = \frac{5\times 8}{\sqrt{6}\times 19}.$$

(The numbers on the exam will be nicer)

(9) The general solution to the equation is always

$$\boldsymbol{x}_{\text{gen}}(t) = \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)$$

where $\boldsymbol{x}_p$ is some particular solution and $\boldsymbol{x}_h$ is the general solution to the homogeneous equation.

The homogeneous equation. Find the eigenvalues/vectors for A. They are

$$\lambda = 4i, \boldsymbol{v} = \begin{bmatrix}1\\-2i\end{bmatrix}, \qquad \bar{\lambda} = -4i, \bar{\boldsymbol{v}} = \begin{bmatrix}1\\2i\end{bmatrix}$$

Therefore the general solution to the homogeneous equation is

$$\boldsymbol{x}_h(t) = c_1 e^{4it} \begin{bmatrix} 1 \\ -2i \end{bmatrix} + c_2 e^{-4it} \begin{bmatrix} 1 \\ 2i \end{bmatrix}.$$

This is the complex form of the general solution. To get the real form you take the real and imaginary parts of one of the complex solutions:

$$\begin{aligned} e^{4it} \begin{bmatrix} 1 \\ -2i \end{bmatrix} &= (\cos 4t + i \sin 4t) \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\} \\ &= \cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix} + i \left\{ \cos 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}. \end{aligned}$$

The real and imaginary parts of a solution to a homogeneous equation are also solutions, so

$$\boldsymbol{x}_1(t) = \cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

and

$$\boldsymbol{x}_2(t) = \cos 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are solutions to $\boldsymbol{x}' - A\boldsymbol{x} = 0$. The general solution in real form is

$$\boldsymbol{x}_h(t) = a_1 \boldsymbol{x}_1(t) + a_2 \boldsymbol{x}_2(t).$$

The inhomogeneous equation. We look for a particular solution of the form $\boldsymbol{x}(t) = e^{i\omega t} \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, and try to find the constants a and b. Substitute in the equation:

$$\boldsymbol{x}' - A\boldsymbol{x} = i\omega e^{i\omega t} \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] - e^{i\omega t} A \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \stackrel{?}{=} e^{i\omega t} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$$

Cancelling exponentials leads to the equations

$$\big(i\omega I - A\big) \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \tag{2}$$

which we can solve for a and b by row reducing

$$\left[\begin{array}{cc|c} i\omega & 2 & 1 \\ -8 & i\omega & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 2 - \frac{1}{8}\omega^2 & 1 \\ 1 & -\frac{1}{8} i\omega & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & \frac{1}{2-\frac{1}{8}\omega^2} \\ 1 & -\frac{1}{8} i\omega & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & \frac{8}{16-\omega^2} \\ 1 & 0 & \frac{i\omega}{16-\omega^2} \end{array}\right]$$

The solution is

$$a = \frac{i\omega}{16 - \omega^2}, \qquad b = \frac{8}{16 - \omega^2}.$$

Therefore the particular solution is

$$\boldsymbol{x}_p(t) = e^{i\omega t} \begin{bmatrix} \frac{i\omega}{16-\omega^2} \\ \frac{8}{16-\omega^2} \end{bmatrix} = \frac{e^{i\omega t}}{16 - \omega^2} \begin{bmatrix} i\omega \\ 8 \end{bmatrix}.$$

This solution is valid as long as $\omega^2 \neq 16$ (i.e. as long as we don't divide by zero), so as long as $\omega \neq \pm 4$.

The general solution. As we said above, the general solution to the inhomogeneous equation is $\boldsymbol{x}_{\text{gen}}(t) = \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)$, where $\boldsymbol{x}_h$ and $\boldsymbol{x}_p$ are the solutions we have just computed.