PRACTICE PROBLEMS FOR THE MATH 320 FINAL, SPRING MMXI

Instructions: For every problem give a short outline of your approach before you compute anything. Then solve the problem according to your plan.

1. For which values of the constant α does the system

$$\begin{aligned} x + 3y - 4z + w &= 1 \\ 2x - 3y - 4z + w &= \alpha \\ -x + 6y &= 1 \end{aligned}$$

have a solution? Find the general solution when there is one.

2. (***i***) State the definition of a *linear subspace* of $\mathbb{R}^n$.

(***ii***) State the definition of *an eigenvector* of a matrix A.

(***iii***) Let S be the set of solutions $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ to $2x_1 + 3x_2 = 1$ i.e., define

$$S = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathbb{R}^2 \mid 2x_1 + 3x_2 = 1 \right\}.$$

Prove or disprove: S is a linear subspace of $\mathbb{R}^2$.

3. For which values of the constant α are the vectors

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} \alpha \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 2\alpha \end{bmatrix},$$

linearly dependent?

4. Find the solution to

$$\boldsymbol{x}''(t) + A\boldsymbol{x} = 0, \text{ where } A = \begin{bmatrix} 7 & -4 \\ -4 & 13 \end{bmatrix}$$

with initial data

$$\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}'(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

5. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 3 & 2 \\ -2 & -3 & 0 & 4 \end{bmatrix}$$

6. For which values of α does the matrix

$$A = \begin{bmatrix} 2 & \alpha \\ 1 & 3+\alpha \end{bmatrix}$$

have only real eigenvalues?

7. Find the eigenvalues and eigenvectors of the matrices

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 5 & 3 & -6 \\ 4 & 2 & -4 \\ 4 & 3 & -5 \end{bmatrix}$$

(or any of the other matrices in problems 1…32 of §6.1, p.374-375)

8. The eigenvalues and vectors of the matrix A are

$$\lambda_1 = -4,\ \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ and } \lambda_2 = -6,\ \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(***i***) If $\boldsymbol{a} = 12\boldsymbol{v}_1 + 24\boldsymbol{v}_2$ then compute the first component of $A^2\boldsymbol{a}$.

(***ii***) Compute $A^{-1}\boldsymbol{a}$ (same $\boldsymbol{a}$ as above).

(***iii***) Use the method of undetermined parameters to solve

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) + \begin{bmatrix} 120 \sin 5t \\ 0 \end{bmatrix}, \text{ with } \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(***iv***) Use the method of undetermined parameters to solve

$$\boldsymbol{x}''(t) = A\boldsymbol{x}(t) + \begin{bmatrix} 120 \sin 5t \\ 0 \end{bmatrix}, \text{ with } \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{x}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

9. Find the general solution to the inhomogeneous system of linear differential equations

$$\boldsymbol{x}' = A\boldsymbol{x} + e^{i\omega t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ where } A = \begin{bmatrix} 0 & -2 \\ 8 & 0 \end{bmatrix} \text{ and } \omega \text{ is arbitrary.}$$

Your solution should be valid for ***almost*** all possible values of the forcing frequency ω: if there are one or two values for which your solution is not valid, indicate those values.