– MATH 320 –
INHOMOGENEOUS LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

1. What to guess

To find a particular solution for a linear inhomogeneous system of differential equations

$$\boldsymbol{x}' - A\boldsymbol{x} = \boldsymbol{f}(t)$$

or of a mechanical system with external force $\boldsymbol{f}(t)$

$$\boldsymbol{x}'' - A\boldsymbol{x} = \boldsymbol{f}(t)$$

you can use the method of undetermined parameters. Here is a short list of recommended guesses for various right hand sides $\boldsymbol{f}(t)$. They work almost always, but there are exceptional situations where they don't. These exceptions occur when the homogeneous equation $\boldsymbol{x}' = A\boldsymbol{x}$ (or $\boldsymbol{x}'' = A\boldsymbol{x}$) has the recommended guess as solution. When this happens "multiplying the guess with t" is often an appropriate remedy.

$\boldsymbol{f}(t)$	$\rightarrow$	$\boldsymbol{x}_p(t)$	
$\boldsymbol{a}$	$\rightarrow$	$\boldsymbol{p}$	
$t^k\boldsymbol{a}$	$\rightarrow$	$P_k(t)\boldsymbol{p}$	$P_k(t)$ a polynomial of degree k
$e^{\alpha t}\boldsymbol{a}$	$\rightarrow$	$e^{\alpha t}\boldsymbol{p}$	
$\sin(\alpha t)\boldsymbol{a} + \cos(\alpha t)\boldsymbol{b}$	$\rightarrow$	$\sin(\alpha t)\boldsymbol{p} + \cos(\alpha t)\boldsymbol{q}$	
$e^{\beta t}\sin(\alpha t)\boldsymbol{a} + e^{\beta t}\cos(\alpha t)\boldsymbol{b}$	$\rightarrow$	$e^{\beta t}\sin(\alpha t)\boldsymbol{p} + e^{\beta t}\cos(\alpha t)\boldsymbol{q}$	

In this table $\boldsymbol{a}, \boldsymbol{b}$ are given constant vectors and α, β are given constant numbers, while $\boldsymbol{p}$ and $\boldsymbol{q}$ are constant vectors which play the role of the undetermined parameter.

Most of the time you will already have computed the eigenvalues and eigenvectors of the matrix A. If you have a basis of eigenvectors of A (which happens in all examples in the homework), then the computations become simpler (ok, less complicated) if, instead of writing the undetermined vectors as $\boldsymbol{p} = \left[\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\right]$. you write them as linear combination of the eigenvectors. Thus if your guess contains an undetermined vector $\boldsymbol{p}$, then write it as

$$\boldsymbol{p} = p_1\boldsymbol{v}_1 + p_2\boldsymbol{v}_2$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2$ are the eigenvectors of A, and solve for p_1 and p_2 (assuming A is a 2×2 matrix; if A is $n \times n$ then you get n terms.)

The advantage of this is that it is now very easy to see the effect of multiplying $\boldsymbol{p}$ with A, namely,

$$A\boldsymbol{p} = p_1 A\boldsymbol{v}_1 + p_2 A\boldsymbol{v}_2 = \lambda_1 p_1 \boldsymbol{v}_1 + \lambda_2 p_2 \boldsymbol{v}_2.$$

2. Problem with an exponential forcing term

Solve

$$\boldsymbol{x}' = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \boldsymbol{x} + e^{\alpha t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{(differential equation)} \tag{1a}$$

$$\boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad \text{(initial condition)} \tag{1b}$$

Note that the right hand side (the so called “forcing term”) has a parameter α in it. We want to find the solution for all values of α.

2.1. **Outline of our computation**. Since the equation is of the form $\boldsymbol{x}' = A\boldsymbol{x} + vf(t)$ the general solution is of the form

$$\boldsymbol{x}_{\text{inh}}(t) = \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t)$$

where $\boldsymbol{x}_h(t)$ is the general solution to the homogeneous equation and $\boldsymbol{x}_p(t)$ is a particular solution to the inhomogeneous equation.

We will first find $\boldsymbol{x}_h(t)$ by computing the eigenvalues&vectors of the matrix A. The solutions will be of the form

$$\boldsymbol{x}_h(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2,$$

where $A\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1$ and $A\boldsymbol{v}_2 = \lambda_2\boldsymbol{v}_2$.

Next, we find a particular solution by “guessing.”

Finally, we adjust the constants which appear in $\boldsymbol{x}_h(t)$ so as to match the initial conditions, i.e. we set $t = 0$ in our general solution $\boldsymbol{x}_{\text{inh}}(t)$ and write out the equations

$$\boldsymbol{x}_h(0) + \boldsymbol{x}_p(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives us two linear equations for the constants c_1, c_2 which appear in $\boldsymbol{x}_h$. Once we have found those we are done.

2.2. **The homogeneous equation**. The homogeneous equation is $\boldsymbol{x}' = A\boldsymbol{x}$ where the matrix $A = \left[\begin{smallmatrix} 3 & 1 \\ 2 & 4 \end{smallmatrix}\right]$ has eigenvalues and vectors

$$\lambda_1 = 2, \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \lambda_2 = 5, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore the solution to the homogeneous equation is

$$\boldsymbol{x}_h(t) = c_1 e^{2t}\boldsymbol{v}_1 + c_2 e^{5t}\boldsymbol{v}_2. \tag{2}$$

In our textbook this is called the *complementary solution.*

2.3. **A particular solution – first attempt**. (If you’re in a hurry, skip this section and go to §1.4; this section is here because it follows the method suggested by the book, but §1.4 gives an easier way of finding the solution.)

Since the right hand side in the equation $\boldsymbol{x}' - A\boldsymbol{x} = \boldsymbol{f}(t)$ is given by

$$\boldsymbol{f}(t) = e^{\alpha t} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

i.e. since it is of the form $e^{\alpha t}\times$ a constant vector, we look for a particular solution of the same form. The simplest formula we could try is

$$\boldsymbol{x}_p(t) = e^{\alpha t}\boldsymbol{a} = e^{\alpha t} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix},$$

for some constant vector $\boldsymbol{a}$. If we substitute this in the left hand side of the equation $\boldsymbol{x}' - A\boldsymbol{x} = \boldsymbol{f}(t)$, then we will (in the end) get a system of linear equation for a_1 and a_2:

$$\boldsymbol{x}' = \alpha e^{\alpha t}\boldsymbol{a}, \quad A\boldsymbol{x} = A\big(e^{\alpha t}\boldsymbol{a}\big) = e^{\alpha t}A\boldsymbol{a},$$

so

$$\boldsymbol{x}' - A\boldsymbol{x} = \alpha e^{\alpha t}\boldsymbol{a} - e^{\alpha t}A\boldsymbol{a} \stackrel{?}{=} e^{\alpha t}\begin{bmatrix}1\\0\end{bmatrix}.$$

Cancel the exponentials:

$$\alpha\boldsymbol{a} - A\boldsymbol{a} = \begin{bmatrix}1\\0\end{bmatrix},$$

which leads to the following system of equations for a_1, a_2

$$\left[\begin{array}{cc|c} \alpha-3 & -1 & 1 \\ -2 & \alpha-4 & 0 \end{array}\right].$$

Row reduction leads to messy algebra with α's:

$$\left[\begin{array}{cc|c} \alpha-3 & -1 & 1 \\ -2 & \alpha-4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2(\alpha-3) & -2 & 2 \\ -2 & \alpha-4 & 0 \end{array}\right]$$

$$\xrightarrow{(\alpha-3)R2+R1} \left[\begin{array}{cc|c} 0 & -2+(\alpha-3)(\alpha-4) & 2 \\ -2 & \alpha-4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & \alpha^2-7\alpha+10 & 2 \\ -2 & \alpha-4 & 0 \end{array}\right]$$

So we find that

$$a_2 = \frac{2}{\alpha^2-7\alpha+10}, \qquad a_1 = \tfrac{\alpha-4}{2}a_2 = \frac{\alpha-4}{\alpha^2-7\alpha+10}$$

The particular solution is therefore

$$\boldsymbol{x}_p(t) = e^{\alpha t}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \frac{e^{\alpha t}}{\alpha^2-7\alpha+10}\begin{bmatrix}\alpha-4\\2\end{bmatrix}.$$

After this the computations get more and more complicated. Instead of going on with this method we follow another approach which exploits the fact that we have already computed the eigenvalues&vectors of A.

2.4. **A particular solution – second attempt.** We will still try a solution of the form $\boldsymbol{x}(t) = e^{\alpha t}\boldsymbol{a}$, but instead of writing the unknown constant vector $\boldsymbol{a}$ as a column vector $\left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right]$, we write it as a linear combination

$$\boldsymbol{a} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2$$

of the two eigenvectors of the matrix A. Thus we try a particular solution of the form

$$\boldsymbol{x}_p(t) = e^{\alpha t}\big\{a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2\big\} = a_1e^{\alpha t}\boldsymbol{v}_1 + a_2e^{\alpha t}\boldsymbol{v}_2$$

To substitute this in the left hand side of the equation $\boldsymbol{x}' - A\boldsymbol{x} = \boldsymbol{f}$ we compute

$$\boldsymbol{x}_p'(t) = \alpha a_1e^{\alpha t}\boldsymbol{v}_1 + \alpha a_2e^{\alpha t}\boldsymbol{v}_2$$

and

$$\begin{aligned} A\boldsymbol{x}_p(t) &= a_1e^{\alpha t}A\boldsymbol{v}_1 + a_2e^{\alpha t}A\boldsymbol{v}_2 \\ &= a_1e^{\alpha t}\lambda_1\boldsymbol{v}_1 + a_2e^{\alpha t}\lambda_2\boldsymbol{v}_2 \\ &= 2a_1e^{\alpha t}\boldsymbol{v}_1 + 5a_2e^{\alpha t}\boldsymbol{v}_2 \end{aligned}$$

where we have used that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are eigenvectors of A with eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$.

Therefore

$$\boldsymbol{x}_p'(t) - A\boldsymbol{x}_p(t) = (\alpha - 2)a_1 e^{\alpha t}\boldsymbol{v}_1 + (\alpha - 5)a_2 e^{\alpha t}\boldsymbol{v}_2. \tag{3}$$

To compare this expression with $\boldsymbol{f}(t)$ we write $\boldsymbol{f}(t)$ as a linear combination of the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$:

$$e^{\alpha t}\begin{bmatrix}1\\0\end{bmatrix} = (\dots)\ \boldsymbol{v}_1 + (\dots)\ \boldsymbol{v}_2$$

We can do this by first writing $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ as a combination of the eigenvectors, and then multiplying with $e^{\alpha t}$. So first we find f_1 and f_2 such that

$$f_1\boldsymbol{v}_1 + f_2\boldsymbol{v}_2 = \begin{bmatrix}1\\0\end{bmatrix}.$$

Since $\boldsymbol{v}_1 = \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$ and $\boldsymbol{v}_2 = \left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ this leads to the following system of equations

$$\begin{array}{c}\begin{matrix} f_1 & f_2 & \end{matrix}\\ \left[\begin{array}{cc|c} 1 & 1 & 1\\ -1 & 2 & 0\end{array}\right]\end{array} \xrightarrow{R1+R2} \left[\begin{array}{cc|c} 1 & 1 & 1\\ 0 & 3 & 1\end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2/3\\ 0 & 1 & 1/3\end{array}\right]$$

so that

$$f_1 = \tfrac{2}{3}, \quad f_2 = \tfrac{1}{3}, \text{ and hence } \begin{bmatrix}1\\0\end{bmatrix} = \tfrac{2}{3}\boldsymbol{v}_1 + \tfrac{1}{3}\boldsymbol{v}_2.$$

Therefore we can write the right hand side $\boldsymbol{f}(t)$ of the differential equation as

$$\boldsymbol{f}(t) = \tfrac{2}{3}e^{\alpha t}\boldsymbol{v}_1 + \tfrac{1}{3}e^{\alpha t}\boldsymbol{v}_1. \tag{4}$$

By combining (3) and (4) we see that $\boldsymbol{x}_p' - A\boldsymbol{x}_p = \boldsymbol{f}(t)$ will hold if

$$(\alpha - 2)a_1 e^{\alpha t}\boldsymbol{v}_1 + (\alpha - 5)a_2 e^{\alpha t}\boldsymbol{v}_2 = \tfrac{2}{3}e^{\alpha t}\boldsymbol{v}_1 + \tfrac{1}{3}e^{\alpha t}\boldsymbol{v}_1$$

holds. Canceling the exponentials $e^{\alpha t}$ on both sides gives us

$$(\alpha - 2)a_1\boldsymbol{v}_1 + (\alpha - 5)a_2\boldsymbol{v}_2 = \tfrac{2}{3}\boldsymbol{v}_1 + \tfrac{1}{3}\boldsymbol{v}_2. \tag{5}$$

Since $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ are linearly independent (*why*?) equation (5) implies

$$(\alpha - 2)a_1 = \tfrac{2}{3} \text{ and } (\alpha - 5)a_2 = \tfrac{1}{3}.$$

From here we find the coefficients a_1, a_2:

$$a_1 = \frac{2/3}{\alpha - 2}, \qquad a_2 = \frac{1/3}{\alpha - 5}.$$

The particular solution we get is

$$\boldsymbol{x}_p(t) = \frac{2}{3}\frac{e^{\alpha t}}{\alpha - 2}\boldsymbol{v}_1 + \frac{1}{3}\frac{e^{\alpha t}}{\alpha - 5}\boldsymbol{v}_2. \tag{6}$$

2.5. **The general solution, and the one with the right initial value**. We can now get the general solution of the diffeq $\boldsymbol{x}' - A\boldsymbol{x} = \boldsymbol{f}$ by adding the particular and the complementary solutions from (6) and (2). Since we have written both $\boldsymbol{x}_p$ and $\boldsymbol{x}_h$ as linear combinations of the eigenvectors, our formula for the general solution will also be such a combination. Here it is: the general solution to (1a) is

$$\boldsymbol{x}(t) = \Big(c_1 e^{2t} + \tfrac{2}{3}\frac{e^{\alpha t}}{\alpha - 2}\Big)\boldsymbol{v}_1 + \Big(c_2 e^{5t} + \tfrac{1}{3}\frac{e^{\alpha t}}{\alpha - 5}\Big)\boldsymbol{v}_2. \tag{7}$$

To find the solution which also satisfies the initial condition (1b) we compute $\boldsymbol{x}(0)$

$$\boldsymbol{x}(0) = \Big(c_1 + \tfrac{2}{3}\frac{1}{\alpha - 2}\Big)\boldsymbol{v}_1 + \Big(c_2 + \tfrac{1}{3}\frac{1}{\alpha - 5}\Big)\boldsymbol{v}_2.$$

Therefore $\boldsymbol{x}(0)=\mathbf{0}$ holds exactly when

$$c_1=-\tfrac{2}{3}\frac{1}{\alpha-2} \text{ and } c_2=-\tfrac{1}{3}\frac{1}{\alpha-5}.$$

Substituting these values of c_1, c_2 in (7), we find that the solution to (1a) and (1b) is

$$\boldsymbol{x}(t)=\tfrac{2}{3}\frac{e^{\alpha t}-e^{2t}}{\alpha-2}\boldsymbol{v}_1+\tfrac{1}{3}\frac{e^{\alpha t}-e^{5t}}{\alpha-5}\boldsymbol{v}_2. \tag{8}$$

3. A problem with a trigonometric right hand side

Find the solution of

$$\boldsymbol{x}'=\begin{bmatrix}-3 & 4\\ 6 & -5\end{bmatrix}\boldsymbol{x}+\begin{bmatrix}\sin t\\ 0\end{bmatrix}, \qquad \boldsymbol{x}(0)=\begin{bmatrix}2\\4\end{bmatrix} \tag{9}$$

3.1. **Our plan.** We first find the eigenvectors/values of the matrix

$$A=\begin{bmatrix}-3 & 4\\ 6 & -5\end{bmatrix}$$

which appears in the problem. this gives us the solution $\boldsymbol{x}_h(t)$ to the homogeneous equation. Then we compute a particular solution $\boldsymbol{x}_p(t)$. The general solution of the problem is then $\boldsymbol{x}_g(t)=\boldsymbol{x}_h(t)+\boldsymbol{x}_p(t)$. This general solution contains two constants. By setting $t=0$ in $\boldsymbol{x}_g(t)$ the initial condition gives us two equations for these constants, which we will then solve.

3.2. **The eigenvalues and vectors.** We have

$$\det(A-\lambda I)=\lambda^2+8\lambda-9=(\lambda+9)(\lambda-1),$$

so the two eigenvalues are

$$\lambda_1=1, \qquad \lambda_2=-9.$$

Solving $(A-\lambda I)\boldsymbol{v}=\mathbf{0}$ for $\lambda=\lambda_1,\lambda_2$ leads us to the following eigenvectors

$$\boldsymbol{v}_1=\begin{bmatrix}1\\1\end{bmatrix}, \qquad \boldsymbol{v}_2=\begin{bmatrix}2\\-3\end{bmatrix}.$$

Therefore the general solution to the homogeneous equation is

$$\boldsymbol{x}_h(t)=c_1e^{t}\boldsymbol{v}_1+c_2e^{-9t}\boldsymbol{v}_2 \tag{10}$$

3.3. **(Partial) Solution 1.** To find a particular solution for $\boldsymbol{x}'-A\boldsymbol{x}=\left[\begin{smallmatrix}\sin t\\0\end{smallmatrix}\right]$ we recognize first that the right hand side has a trigonometric form

$$\begin{bmatrix}\sin t\\ 0\end{bmatrix}=\sin(t)\,\boldsymbol{a}+\cos(t)\,\boldsymbol{b}$$

for certain constant vectors $\boldsymbol{a},\boldsymbol{b}$. In fact,

$$\boldsymbol{a}=\begin{bmatrix}1\\0\end{bmatrix}, \qquad \boldsymbol{b}=\begin{bmatrix}0\\0\end{bmatrix}.$$

In this case you can (almost) always find a particular solution of the same form, so we will set

$$\boldsymbol{x}_p(t)=\sin(t)\,\boldsymbol{p}+\cos(t)\,\boldsymbol{q}=\begin{bmatrix}p_1\sin t+q_1\cos t\\ p_2\sin t+q_2\cos t\end{bmatrix}$$

and substitute in the equation. Diligent computation then leads you to

$$\boldsymbol{x}_p' = \begin{bmatrix} -q_1 \sin t + p_1 \cos t \\ -q_2 \sin t + p_2 \cos t \end{bmatrix}$$

$$A\boldsymbol{x}_p = \begin{bmatrix} (-3p_1 + 4p_2) \sin t + (-3q_1 + 4q_2) \cos t \\ (6p_1 - 5p_2) \sin t + (6q_1 - 5q_2) \cos t \end{bmatrix}$$

so that

$$\boldsymbol{x}_p' - A\boldsymbol{x}_p = \begin{bmatrix} (3p_1 - 4p_2 - q_1) \sin t + (p_1 + 3q_1 - 4q_2) \cos t \\ (-6p_1 + 5p_2 - q_2) \sin t + (p_2 - 6q_1 + 5q_2) \cos t \end{bmatrix}.$$

The equation $\boldsymbol{x}_p' - A\boldsymbol{x}_p = \left[\begin{smallmatrix} \sin t \\ 0 \end{smallmatrix}\right]$ then leads to the following four equations for p_1, p_2, q_1, q_2:

$$\left[\begin{array}{cccc|c} p_1 & p_2 & q_1 & q_2 & \\ 3 & -4 & -1 & 0 & 1 \\ 1 & 0 & 3 & -4 & 0 \\ -6 & 5 & 0 & -1 & 0 \\ 0 & 1 & -6 & 5 & 0 \end{array}\right]$$

3.4. **Second approach – use the eigenvectors of** A. Look for a particular solution in the form

$$\boldsymbol{x}_p(t) = c_1(t)\boldsymbol{v}_1 + c_2(t)\boldsymbol{v}_2,$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2$ are the eigenvectors of the matrix A above. Since the right hand side of the equation contains the function $\sin t$ we will let the coefficients $c_1(t)$ and $c_2(t)$ be similar trigonometric expressions. Therefore we will try

$$\boldsymbol{x}_p(t) = \big(P \sin t + Q \cos t\big)\boldsymbol{v}_1 + \big(R \sin t + S \cos t\big)\boldsymbol{v}_2.$$

Substituting in the left hand side of the equation leads to

$$\boldsymbol{x}_p' = \big(-Q \sin t + P \cos t\big)\boldsymbol{v}_1 + \big(-S \sin t + R \cos t\big)\boldsymbol{v}_2,$$

and

$$\begin{aligned} \boldsymbol{x}_p(t) &= \big(P \sin t + Q \cos t\big)A\boldsymbol{v}_1 + \big(R \sin t + S \cos t\big)A\boldsymbol{v}_2 \\ &= \big(P \sin t + Q \cos t\big)\boldsymbol{v}_1 + \big(-9R \sin t - 9S \cos t\big)\boldsymbol{v}_2, \end{aligned}$$

since $A\boldsymbol{v}_1 = \boldsymbol{v}_1$ and $A\boldsymbol{v}_2 = -9\boldsymbol{v}_2$. Therefore

$$\begin{aligned} \boldsymbol{x}_p' - A\boldsymbol{x}_p = &\big\{(-Q - P) \sin t + (P - Q) \cos t\big\}\boldsymbol{v}_1 \\ &+ \big\{(-S + 9R) \sin t + (R + 9S) \cos t\big\}\boldsymbol{v}_2. \end{aligned} \tag{11}$$

On the other side of the equation $\boldsymbol{x}' - A\boldsymbol{x} = \boldsymbol{f}(t)$ we have

$$\boldsymbol{f}(t) = \begin{bmatrix} \sin t \\ 0 \end{bmatrix} = \sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

To compare this with our expression (11) for $\boldsymbol{x}_p' - A\boldsymbol{x}_p$, we write $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ as a combination of the eigenvectors:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = f_1\boldsymbol{v}_1 + f_2\boldsymbol{v}_2,$$

Using $\boldsymbol{v} - 1 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\boldsymbol{v}_2 = \left[\begin{smallmatrix} 2 \\ -3 \end{smallmatrix}\right]$ this leads to the equations

$$\left[\begin{array}{cc|c} f_1 & f_2 & \\ 1 & 2 & 1 \\ 1 & -3 & 0 \end{array}\right] \xrightarrow{-R1+R2} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3/5 \\ 0 & 1 & 1/5 \end{array}\right]$$

so that we have

$$\boldsymbol{f}(t) = \sin(t)\begin{bmatrix}1\\0\end{bmatrix} = \tfrac{3}{5}\sin(t)\boldsymbol{v}_1 + \tfrac{1}{5}\sin(t)\boldsymbol{v}_2.$$

Comparing with (11) we get the following equations for P, Q, R, S

$$\begin{matrix} -Q - P = \frac{3}{5} \\ P - Q = 0 \end{matrix}, \qquad \begin{matrix} -S + 9R = \frac{1}{5} \\ R + 9S = 0 \end{matrix}$$

Solving these equations gives us

$$P = Q = \tfrac{3}{10}, \text{ and } R = \tfrac{9}{410},\ S = -\tfrac{1}{410}.$$

We therefore get this particular solution:

$$\boldsymbol{x}_p(t) = \left(\tfrac{3}{10}\sin t + \tfrac{3}{10}\cos t\right)\boldsymbol{v}_1 + \left(\tfrac{9}{410}\sin t - \tfrac{1}{410}\cos t\right)\boldsymbol{v}_2.$$

3.5. **The general solution, and the one that satisfies the initial conditions**. Since the complementary solution we have found is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, it is easy to add $\boldsymbol{x}_h$ and $\boldsymbol{x}_p$ to get the general solution to the equation. We get

$$\begin{aligned}\boldsymbol{x}_g(t) &= \boldsymbol{x}_h(t) + \boldsymbol{x}_p(t) \\ &= \left(c_1 e^t + \tfrac{3}{10}\sin t + \tfrac{3}{10}\cos t\right)\boldsymbol{v}_1 + \left(c_2 e^{-9t} + \tfrac{9}{410}\sin t - \tfrac{1}{410}\cos t\right)\boldsymbol{v}_2.\end{aligned}$$

Finally, we can find the solution which satisfies the initial condition $\boldsymbol{x}(0) = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ from (9) by computing

$$\boldsymbol{x}_g(0) = \left(c_1 + \tfrac{3}{10}\right)\boldsymbol{v}_1 + \left(c_2 - \tfrac{1}{410}\right)\boldsymbol{v}_2.$$

To compare this with $\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ we write this vector as a combination of the eigenvectors:

$$\begin{bmatrix}2\\4\end{bmatrix} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2$$

leads to

$$\left[\begin{array}{cc|c} \overset{a_1}{1} & \overset{a_2}{2} & 2 \\ 1 & -3 & 4 \end{array}\right] \xrightarrow{-R1+R2} \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & -5 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2\,{}^4\!/_5 \\ 0 & 1 & -{}^2\!/_5 \end{array}\right]$$

so that

$$\begin{bmatrix}2\\4\end{bmatrix} = \tfrac{14}{5}\boldsymbol{v}_1 - \tfrac{2}{5}\boldsymbol{v}_2.$$

The initial condition $\boldsymbol{x}_g(0) = \left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ therefore implies

$$c_1 + \tfrac{3}{10} = \tfrac{14}{5} \text{ and } c_2 - \tfrac{1}{410} = -\tfrac{2}{5},$$

so

$$c_1 = \tfrac{31}{10} \qquad c_2 = -\tfrac{163}{410}.$$

The solution which satisfies the initial conditions is

$$\boldsymbol{x}(t) = \left(\tfrac{31}{10}e^t + \tfrac{3}{10}\sin t + \tfrac{3}{10}\cos t\right)\boldsymbol{v}_1 + \left(-\tfrac{163}{410}e^{-9t} + \tfrac{9}{410}\sin t - \tfrac{1}{410}\cos t\right)\boldsymbol{v}_2.$$