math 320 — the second midterm

Friday April 1st, 2011

1. (a) [5 points] *Complete the sentence:*

- *According to the definition, the vectors* $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ *are **linearly independent** if...*

  the only solution to $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$.

(b) [5 points] *Complete the sentence:*

- *Suppose* L *is a linear subspace of* $\mathbb{R}^k$ *and suppose* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *are vectors belonging to* L. *According to the definition **a set of vectors*** $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ ***span*** L ***if...***

  every vector in L is a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$.

(c) [5 points] *Are the vectors*

$$\mathbf{u} = \begin{bmatrix} 1 \\ 11 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 12 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} -2 \\ 1 \\ 9 \end{bmatrix}$$

linearly independent? (Explain your answer).

They are linearly dependent. Possible justifications for this conclusion:

1. $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ are vectors in $\mathbb{R}^3$. Since $\mathbb{R}^3$ is three dimensional, you can't have more than three independent vectors in $\mathbb{R}^3$.

2. If you try to solve $c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w}+c_4\mathbf{z} = 0$ then you get three linear equations with four unknowns. Since there are more unknowns than equations there will always be a solution other than $c_1 = c_2 = c_3 = c_4 = 0$.

(Some actually wrote out the system of equations for $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_4\mathbf{z} = 0$ and then tried to solve it. This is a correct approach, although it is too much work because you know ahead of time that the solutions will contain a parameter. Therefore there will be a nontrivial solution.)

(d) [5 points] *Are the vectors*

$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}$$

linearly independent? (Explain your answer).

The system of equations $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ has the following matrix associated with it,

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

You can row reduce this:

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[(-2)R_2+R_3]{\substack{R_2/3 \\ (-1)R_2+R_1}} \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[(-3)R_1+R_4]{\substack{R_1/2 \\ (-1)R_1+R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

after which you see that $c_1 = c_2 = c_3 = 0$ is the only solution.

Alternatively, you could also argue as follows: The second equation is $3c_2 = 0$, so $c_2 = 0$. The first equation then implies that $c_1 = -c_3$. Substitute this in the fourth equation and you find that $c_3 = 0$, and hence also that $c_1 = 0$.

Either way, the conclusion is that these vectors are linearly independent.

2. (a) [10 points] *Use row reduction to find the general solution of this system of equations:*

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + 3x_4 &= 0 \\ 3x_1 + 4x_2 + 2x_3 + 5x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ 3 & 4 & 2 & 5 & 0 \end{array}\right] \xrightarrow{\substack{-R_1+R_2 \\ -3R_1+R_3}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\substack{-R_2+R_3 \\ -R_2+R_1}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the general solution is given by

$$\mathbf{x} = \begin{bmatrix} -2s+t \\ s-2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(b)[5 points] *Let S be the solution space for the equations in part (a) of this problem. Find a basis for S.*

From our formula for the general solution to the system of equations we get a basis for the solutions space namely

$$\{\mathbf{v}, \mathbf{w}\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c)[5 points] *What is the dimension of the solution space S? Explain your answer.*

We have a basis for the solution space. This basis consists of two vectors. Therefore the dimension of the solution space is **two**.

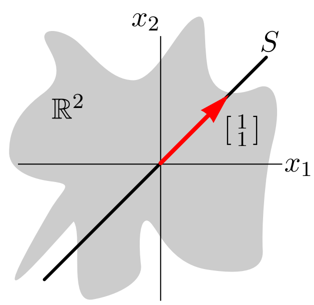
Many were confused about the concept "dimension" and said that the vectors have four entries, so the solutions space is four dimensional. This is wrong. To explain, consider the simpler example where S is the solution space in $\mathbb{R}^2$ of the equation

$$x_1 - x_2 = 0.$$

The solution space consists of all vectors of the form

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ is a basis for S. In this example we can draw the solution space: it is the diagonal $x_1 = x_2$ in the plane $\mathbb{R}^2$. Here all vectors in S have two components, but together they fill up the line S, which is **one-dimensional**.



In the same way the solution space S from the exam problem sits in a four dimensional space ($\mathbb{R}^4$), but S itself is two dimensional.

3. Let $y(t)$ be the solution of the differential equation

$$\frac{dy}{dt} = (1 - 2t)y$$

with initial condition $y(0) = 5.0$. (Clever people have found that the solution is $y(t) = 5e^{t-t^2}$, but you do no need this formula in this problem.)

(a) [8 points] *Explain how one step of Euler's method allows you to find an approximation of $y(t + \Delta t)$ if you know $y(t)$. (State, but do not derive, what Euler's method asks you to compute.)*

Here the answer could be given in many different ways, but in the end it must be clear from what you wrote how Euler tells you to compute $y(t + \Delta t)$ from $y(t)$. If you use y_n and y_{n+1} then you have to explain how they are related to $y(t)$ and $y(t + \Delta t)$.

A possible solution would be:

> "Given $y(t)$, the differential equation tells you what $y'(t)$ is;
> Euler then says that $y(t + \Delta t) \approx y(t) + \Delta t \times y'(t)$."

Or:

> "If you know $y(t)$, and if y is a solution of the differential equation $y' = f(t, y)$, then you can compute $y'(t) = f(t, y(t))$, and Euler says that $y(t + \Delta t) \approx y(t) + \Delta t\ f(t, y(t))$."

(b) [5 points] *Use Euler's method to approximate $y(0.2)$, using step size $\Delta t = 0.2$.*

If $\Delta t = 0.2$, and if you want to approximate $y(0.2)$ when you know $y(0)$, then you only have to do one step of Euler's method.

$$y'(0) = (1 - 2 \times 0) \times y(0) = 5 \implies y(0.2) \approx y(0) + \Delta t \times y'(0) = 5 + 0.2 \times 5 = 6.0$$

(c) [4 points] *Use Euler's method to approximate $y(0.2)$, using step size $\Delta t = 0.1$.*

If $\Delta t = 0.1$, and if you want to approximate $y(0.2)$ when you know $y(0)$, then you only have to do two steps of Euler's method.

$y'(0) = (1 - 2 \times 0) \times y(0) = 5 \implies y(0.1) \approx y(0) + \Delta t \times y'(0) = 5 + 0.1 \times 5 = 5.5$

$y'(0.1) = (1-2\times 0.1)\times y(0.1) \approx 0.8\times 5.5 = 4.4 \implies y(0.2) \approx y(0.1)+\Delta t\times y'(0.1) \approx 5.5+0.1\times 4.4 = 5.5 + 0.44 = 5.94$.

(d) [3 points] *Which of the two approximations above do you expect to be closer to the exact value of $y(0.2)$? (Assuming you made no arithmetic mistakes).*

Generally, reducing the step size increases the accuracy of the approximation provided by Euler's method. So the answer from (c) should be considered more reliable.

This is a rule of the thumb, and is certainly not always true.

4. Consider the differential equation

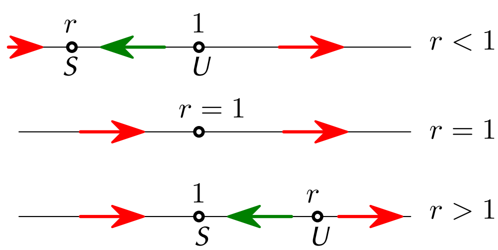
$$\frac{dx}{dt} = (r - x)(1 - x)$$

in which r is a parameter .

(a) [10 points] *Find all equilibrium solutions and draw the phase portraits for the differential equation (if the phase portrait depends on r, then you may have to draw more than one).*

An equilibrium value is a value x for which the differential equation prescribes $\frac{dx}{dt} = 0$. Thus you find the equilibrium solutions by solving $(r - x)(1 - x) = 0$. There are two solutions, namely $x = r$ and $x = 1$.

To see which of these equilibria is stable and which is unstable, we check how the sign of $\frac{dx}{dt}$ changes with x. Since $\frac{dx}{dt} = (r - x)(1 - x)$ we see that $\frac{dx}{dt} < 0$ when x lies between 1 and r, while $\frac{dx}{dt} < 0$ when x lies outside of this interval. This gives us the following phase diagrams:



There are three possibilities, depending on whether $r > 1$, $r < 1$ or $r = 1$.

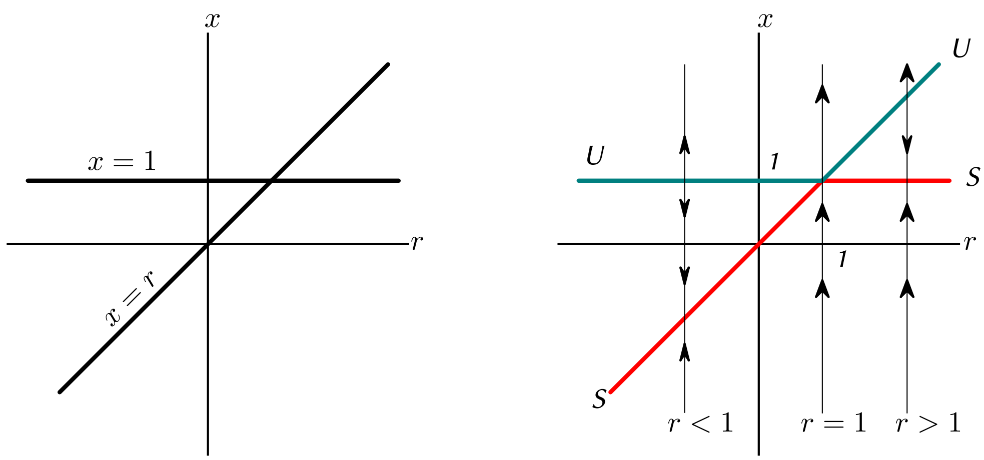
When $r < 1$, the equilibrium $x = 1$ is unstable, and $x = r$ is stable.

When $r > 1$, the equilibrium $x = 1$ is stable, and $x = r$ is unstable.

When $r = 1$, there is only one equilibrium, namely $x = 1$. This equilibrium is neither stable no unstable (or you could say it is stable from the side $x < 1$ but unstable on the side $x > 1$).

(b) [10 points] *Then draw the bifurcation diagram, indicating which solutions are stable, and which are unstable.*

The bifurcation diagram displays how the equilibria of the differential equation depend on the parameter r. The parameter r appears on the horizontal axis, the unknown variable x of the diffeq is on the vertical axis. Each vertical line in the bifurcation diagram corresponds to a certain value of r. On this line we can indicate the sign of $\frac{dx}{dt}$ by placing arrows.



The bifurcation diagram: on the left the diagram with only the equilibria $x=1$ and $x=r$;
on the right the same diagram with the signs of $\frac{dx}{dt}$ and the stability of the equilibria included.

5. [20 points] There is a homogeneous linear differential equation of order 3, for which someone has found that

$$y_1(x)=e^x, \quad y_2(x)=xe^x, \quad y_3(x)=1,$$

are solutions. The superposition principle implies that

$$(\dagger) \qquad y(x)=c_1e^x+c_2xe^x+c_3$$

is a solution for any choice of constants c_1, c_2, c_3.

Does equation ($\dagger$) *provide the general solution? Explain your answer: first explain what you would compute to decide if $y(x)$ in equation* ($\dagger$) *is the general solution, then execute your plan.*

We are not told what the differential equation is, but we are given that it is of order three. Therefore the expression $y(x)=c_1e^x+c_2xe^x+c_3$ is the general solution of the differential equation if for any "initial point x_0" and for any choice of constants a_0, a_1, a_2 you can find constants c_1, c_2, c_3 such that $y(x)$ satisfies the initial conditions

$$(\ddagger) \qquad y(x_0)=a_0, \quad y'(x_0)=a_1, \quad y''(x_0)=a_2.$$

The relevance of the fact that the differential equation is of third order is that there should be ***three*** initial values, and thus ***three*** constants $c_{\dots}$ in the general solution.

When you substitute the formula in (†) for $y(x)$ in the initial conditions (‡), you get a system of three linear equations for c_1, c_2, c_3. This system is always solvable if and only the determinant of coefficients is nonzero. This leads you to the quick test whether one can satisfy the initial conditions for any choice of a_0, a_1, a_2. Namely, compute the Wronskian: if $W(x_0)\neq 0$, then y is indeed the general solution.

A correct solution to this problem would be:

Plan: I will compute the Wronskian and check if it is zero or not.

The Wronskian is

$$\begin{aligned} W(x) &= \begin{vmatrix} e^x & xe^x & 1 \\ e^x & xe^x + e^x & 0 \\ e^x & xe^x + 2e^x & 0 \end{vmatrix} \\ &= \begin{vmatrix} e^x & xe^x + e^x \\ e^x & xe^x + 2e^x \end{vmatrix} \qquad \text{(subtract top from bottom rows)} \\ &= \begin{vmatrix} e^x & xe^x + e^x \\ 0 & e^x \end{vmatrix} \\ &= e^{2x}. \end{aligned}$$

Thus the Wronskian is nowhere zero, and hence $y(x)$ is the general solution.