Math 320, "spring" 2011
– before the first midterm –

Typical Exam Problems

1. Consider the linear system of equations

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + x_4 &= y_1 \\ x_1 + 3x_2 - 2x_3 + 2x_4 &= y_2 \\ x_1 + 2x_3 - x_4 &= y_3 \end{aligned}$$

where $x_1, \cdots, x_4$ are the unknowns, and y_1, y_2, y_3 are given constants.
(***i***) If you want to find the general solution to this system by row reduction, then which matrix do you have to row-reduce?
(***ii***) Compute the Reduced Row Echelon Form of the matrix you found in (***i***).
(***iii***) What is the general solution to the system of equations?
(***iv***) Answer the same questions for the system

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + x_4 &= y_1 \\ x_1 + 3x_2 - 2x_3 + 2x_4 &= y_2 \\ 2x_1 + 2x_3 - x_4 &= y_3 \end{aligned}$$

2. A system of two equations with four unknowns has been row-reduced to the matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 2 & 3 & 1 \end{array}\right].$$

Find the general solution to this system of equations.

3. Consider the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(***i***) Which system of equations does this matrix stand for?
(***ii***) Find the general solution of that system of equations.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & k \\ 0 & 1 & 3 \end{bmatrix}$$

in which k is a constant.
(***i***) For which values of the constant k does the matrix have an inverse?
(***ii***) Show how you use row reduction to find the inverse of A if $k = 0$.

5. Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}.$$

(***i***) Compute the determinants of these matrices.
(***ii***) Compute $\det A^{25}$.
(***iii***) Compute $\det(A + B)$.
(***iv***) Compute $\det(A^2BAB^2)$.
(***v***) If C is a 2×2 matrix with $\det C = +2$, compute $\det(-C)$ and $\det(C^{-1})$.

Inverses and determinants of square matrices

Here is a brief summary of some of the concepts that are introduced in the book. IT IS ASSUMED THAT YOU HAVE READ THE BOOK. After the summary there are also solutions to some of the "conceptual problems" in the homework. Answers to the other homework problems can be found in the book.

Matrix Operations (§3.4)

You can add and multiply matrices, and just as with ordinary numbers matrix addition & multiplication is *associative*:

$$(A+B)+C = A+(B+C), \qquad A(BC) = (AB)C,$$

and *distributive*:

$$A(B+C) = AB + AC, \qquad (A+B)C = AC + BC$$

but it is normally not *commutative*, i.e. for most matrices A and B

$$AB \neq BA.$$

In those special cases where $AB = BA$, the matrices A and B are said to commute: "A and B commute."

Inverses (§3.5)

Three equivalent definitions of A^{-1} for a square matrix A

Suppose we are given two $n \times n$ matrices

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}.$$

First description of A^{-1}

We say $B = A^{-1}$ if the solution of the system of equations

$$\boxed{\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= y_1 \\ \vdots \qquad &= \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= y_n \end{aligned}} \quad \text{is given by} \quad \boxed{\begin{aligned} x_1 &= b_{11}y_1 + \cdots + b_{1n}y_n \\ \vdots &= \qquad \vdots \\ x_n &= b_{n1}y_1 + \cdots + b_{nn}y_n \end{aligned}}$$

Second description of A^{-1}

We say $B = A^{-1}$ if

$$\boxed{\text{the solution of the system of equations } Ax = y \text{ is always given by } x = By}$$

where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

are column vectors.

Note that this second description is just the same as the first, but written in matrix form, using matrix multiplication to interpret Ax and By.

Third description of A^{-1}

We say $B = A^{-1}$ if
$$AB = BA = I.$$

The inverse of a product

If A and B are invertible, then so is AB and one has
$$(AB)^{-1} = B^{-1}A^{-1}.$$
Note the reversal of order. This is important since with matrices AB usually is not equal to BA.

Determinants (§3.6)

Basic properties that help in computing determinants

- You can factor out a constant from any row or column in a determinant, e.g.
$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ ca_{21} & \cdots & ca_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = c \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$
- Adding a multiple of any row to another row in a determinant does not change its value. The same applies to columns. Example:
$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} + ka_{11} & \cdots & a_{2n} + ka_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$
- Swapping two rows changes the sign of a determinant.

Important properties of determinants

$$\begin{aligned} \det(A^T) &= \det A \\ \det(AB) &= \det(A)\det(B) \\ \det(A^{-1}) &= \frac{1}{\det A} \qquad \text{(provided } \det A \neq 0\text{)} \end{aligned}$$

Solutions to some problems

§3.4 – Problem 31

You are asked to show that $(A + B)(A - B) \neq A^2 - B^2$, for the two matrices A and B from example 5. You could look up what A and B are and compute both $(A + B)(A - B)$ and $A^2 - B^2$ (the answers are in the back of the book), but this misses the point somewhat. Instead, you learn more by doing the following calculation:

$$\begin{aligned}(A-B)(A+B) &= A(A+B) - B(A+B)\\ &= AA + AB - BA - BB\\ &= A^2 - B^2 + AB - BA.\end{aligned}$$

So if $AB = BA$ then $(A-B)(A+B) = A^2 - B^2$ is true, but if $AB \neq BA$, then it is not true.

For A and B as in example 5, one has $AB \neq BA$ and therefore $(A+B)(A-B) \neq A^2 - B^2$.

§*3.4 – Problem 32*

The same comments as for the previous problem apply to this problem. The most instructive solution is as follows:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A(A+B) + B(A+B)\\ &= AA + AB + BA + BB\\ &= A^2 + AB + BA + B^2.\end{aligned}$$

So if $AB = BA$ then $(A+B)^2 = A^2 + 2AB + B^2$ is true (because $AB + BA = AB + AB = 2AB$), but if $AB \neq BA$, then it is not true.

§*3.5 – Problem 32*

If A is invertible, and if $AB = AC$, then multiply both sides of the equation from the left with A^{-1}, and you find

$$AB = AC \implies A^{-1}AB = A^{-1}AC \implies IB = IC \implies B = C.$$

§*3.5 – Problem 34*

A diagonal matrix is a matrix of the form

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0\\ 0 & d_2 & 0 & \cdots & 0\\ \vdots & & & & \cdots\\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

Since none of the diagonal entries $d_1, d_2, \cdots, d_n$ are zero (that is the assumption in the problem), you can define

$$E = \begin{bmatrix} 1/d_1 & 0 & 0 & \cdots & 0\\ 0 & 1/d_2 & 0 & \cdots & 0\\ \vdots & & & & \cdots\\ 0 & 0 & 0 & \cdots & 1/d_n \end{bmatrix}.$$

By computing the following matrix products you find

$$ED = I, \text{ and also } DE = I.$$

Therefore D is invertible and its inverse is given by E.

§3.6 – Problem 50

If a matrix A satisfies $A^2 = A$, then

$$\begin{aligned} & \det(A^2) = \det(A) \\ \Longrightarrow\ & \det(A \cdot A) = \det(A) \\ \Longrightarrow\ & \det(A) \cdot \det(A) = \det(A) \\ \Longrightarrow\ & \det(A)^2 = \det(A). \end{aligned}$$

Therefore $\det(A)$ is a number which satisfies $(\det A)^2 = \det A$. This implies $\det A = 0$ or $\det A = 1$.

(More detail: if $x = \det A$, then we have shown $x^2 = x$. This is an equation for x. Either $x = 0$, or else you can divide both sides by x, in which case you find $x = 1$.)

§3.6 – Problem 51

If a matrix A satisfies $A^n = \mathbf{0}$ forsome integer n, then show that $\det A = 0$.

Solution. Using the basic property $\det(AB) = \det(A)\det(B)$ we find

$$\det(A^n) = \det(A \cdot A \cdots A) = \det(A) \cdot \det(A) \cdots \det(A) = \big(\det(A)\big)^n.$$

Therefore, if $\det(A^n) = 0$, then $(\det A)^n = 0$. Since $\det A$ is a number $(\det A)^n$ implies $\det A = 0$.

§3.6 – Problem 52

The problem tells you that certain matrices are called "orthogonal." Ignoring the terminology, the problem asks this: *if a matrix A satisfies $A^T = A^{-1}$, then show that its determinant is either $+1$ or -1.*

Solution. Since $\det A^T = \det A$, and since $\det A^{-1} = 1/\det A$, we find that if $A^T = A^{-1}$, then the determinant of A must satisfy

$$\det A = \frac{1}{\det A},$$

Multiply both sides with $\det A$ and you find that $(\det A)^2 = 1$. Therefore $\det A = \pm 1$.

§3.6 – Problem 53

Again some terminology is introduced which, for the purposes of solving the problem, you can ignore.

The problem asks you to show that if three square matrices A, B, P are related by $A = P^{-1}BP$, then A and B have the same determinant.

Solution. Use the properties of the determinant:

$$\det(A) = \det(P^{-1}BP) = (\det P^{-1})(\det B)(\det P) = \frac{1}{\det P}(\det B)(\det P) = \det B.$$

That's all. Note that in the last step you can cancel the two $\det P$'s, because they are numbers. When you look at the expression $P^{-1}BP$ you might think that you can cancel P and P^{-1} because $P^{-1}P = I$. But you can't because in the product

$P^{-1}BP$ there's a B between the P^{-1} and P. **If** it were true that $BP = PB$, **then** you could say

$$\begin{aligned} P^{-1}BP &= P^{-1}PB && \text{because } BP = PB \\ &= IB && \text{because } P^{-1}P = I \\ &= B, \end{aligned}$$

but if $BP \neq PB$ then this doesn't work.

§3.6 – Problem 54

You are asked to prove:

$$AB \text{ is invertible} \iff \text{both } A \text{ and } B \text{ are invertible.}$$

Solution. This is the section about determinants, and one of the main uses of determinants is that they tell you when a matrix is invertible:

$$A \text{ is invertible} \iff \det A \neq 0.$$

Apply this to the product AB:

$$AB \text{ is invertible} \iff \det AB \neq 0.$$

Since $\det AB = (\det A)(\det B)$, we get

$$AB \text{ is invertible} \iff (\det A)(\det B) \neq 0.$$

The product of two numbers (such as $\det A$ and $\det B$) is nonzero if and only if both numbers are nonzero, so we find

$$AB \text{ is invertible} \iff \det A \neq 0 \text{ and } \det B \neq 0.$$

Finally, $\det A \neq 0$ is equivalent to "A is invertible," and the same for B. Therefore

$$AB \text{ is invertible} \iff A \text{ is invertible and } B \text{ is invertible}$$

§3.6 – Problem 56

If all entries of a matrix A are integers, then its determinant also is an integer because you find it by adding and multiplying entries of A (you never have to divide.)

If $B = A^{-1}$ then the ij entry of B is given by Cramer's formula

$$b_{ij} = \frac{A_{ji}}{\det A}.$$

Here A_{ji} is the ji cofactor of the matrix A. It is an $(n-1)\times(n-1)$ determinant whose entries are integers because they come from A. Therefore A_{ji} is an integer. If we are given that $\det A = 1$, then our formula for b_{ij} shows us that

$$b_{ij} = \frac{A_{ji}}{\det A} = A_{ji},$$

In particular, b_{ij} is an integer.

Answers and Hints

(1i) There are two options. Either you keep the constants y_1, y_2, y_3, in which case you get

$$\begin{array}{cccc|c} 2 & 3 & -2 & 1 & y_1 \\ 1 & 3 & -2 & 2 & y_2 \\ 1 & 0 & 2 & -1 & y_3 \end{array}$$

or you don't write the constants y_1, y_2, y_3 but you keep track of their coefficients instead. Then you get the bigger matrix

$$\begin{array}{cccc|ccc} 2 & 3 & -2 & 1 & 1 & 0 & 0 \\ 1 & 3 & -2 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}$$

(1ii) I don't write the coefficients of y_1, y_2, y_3 and work with the bigger matrix which only contains coefficients instead. This leads me to the RREF:

$$\ldots \xrightarrow{(-R2)+R1} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 3 & -2 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-R1)+R3]{(-R1)+R2} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 3 & -2 & 3 & -1 & 2 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{(R3)+R2} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 3 & 0 & 3 & -2 & 3 & 1 \\ 0 & 0 & 2 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R2/3, R3/2} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2/3 & 1 & 1/3 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \end{array}\right].$$

If you had carried the y_1, y_2, y_3 along in your computation, you would have ended up with

$$\left[\begin{array}{cccc|l} 1 & 0 & 0 & -1 & y_1 - y_2 \\ 0 & 1 & 0 & 1 & -\frac{2}{3}y_1 + y_2 + \frac{1}{3}y_3 \\ 0 & 0 & 1 & 0 & -\frac{1}{2}y_1 + \frac{1}{2}y_2 + \frac{1}{2}y_3 \end{array}\right].$$

(1iii) The solution has one parameter in it, since the RREF does not tell you what x_4 is. If we call the parameter t then we get the following solution

$$\begin{aligned} x_1 &= y_1 - y_2 + t; \\ x_2 &= -\tfrac{2}{3}y_1 + y_2 + \tfrac{1}{3}y_3 - t \\ x_3 &= -\tfrac{1}{2}y_1 + \tfrac{1}{2}y_2 + \tfrac{1}{2}y_3 \\ x_4 &= t. \end{aligned}$$

(1iv)

$$\left[\begin{array}{cccc|ccc} 2 & 3 & -2 & 1 & 1 & 0 & 0 \\ 1 & 3 & -2 & 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-R2)+R1} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 3 & -2 & 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-2R1)+R3]{(-R1)+R2} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 3 & -2 & 3 & -1 & 2 & 0 \\ 0 & 0 & 2 & +1 & -2 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{(-R3)+R2} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 3 & 0 & 4 & -3 & 4 & 1 \\ 0 & 0 & 2 & +1 & -2 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R2/3, R3/2} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4/3 & -1 & 4/3 & 1/3 \\ 0 & 0 & 1 & 1/2 & -1 & 1 & 1/2 \end{array}\right].$$

The general solution is

$$\begin{aligned} x_1 &= y_1 - y_2 + t \\ x_2 &= -y_1 - \tfrac{4}{3}y_2 + \tfrac{1}{3}y_3 - \tfrac{4}{3}t \\ x_3 &= -y_1 + y_2 + \tfrac{1}{2}y_3 + \tfrac{1}{2}t \\ x_4 &= t \end{aligned}$$

(2) The matrix is already in reduced row echelon form. You can solve for x_1 and x_2, but x_3 and x_4 are undetermined, so we make them parameters, s and t. The general solution is then

$$\begin{aligned} x_1 &= 4 - t \\ x_2 &= 1 - 2s - 3t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

(3i)

$$\begin{array}{ccccc} x_1 & & +3x_3 & = 4 \\ & x_2 & & = 0 \\ & & 0 & = 1 \end{array}$$

(3ii) The last equation $0 = 1$ is not satisfied for any choice of $x - 1, x_2, x_3$, so there is no solution.

(4i) Compute the determinant of the matrix to see if it has an inverse.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & k \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & k-2 \\ 0 & 1 & 3 \end{vmatrix} = -(k-2)\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -k + 2.$$

(in the first step we subtract the first row twice from the second row.) So if $k \neq 2$ the matrix has an inverse because its determinant is nonzero.

(4ii) Here is how you get the inverse for any k except $k = 2$. The question asked you to do this with $k = 0$.

Form the matrix $[A|I]$ which you get from A by appending the identity matrix. Then row reduce until you get $[I|B]$. The matrix B is the inverse. Here is one way to get there:

$$\left.\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & k & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right. \longrightarrow \left.\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & k-2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right.$$

$$\longrightarrow \left.\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & k-2 & -2 & 1 & 0 \end{array}\right.$$

$$\longrightarrow \left.\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & k-2 & -2 & 1 & 0 \end{array}\right.$$

$$\longrightarrow \left.\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2/(k-2) & 1/(k-2) & 0 \end{array}\right.$$

$$\longrightarrow \left.\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-10/(k-2) & 5/(k-2) & -2 \\ 0 & 1 & 0 & 6/(k-2) & -3/(k-2) & 1 \\ 0 & 0 & 1 & -2/(k-2) & 1/(k-2) & 0 \end{array}\right.$$

Therefore the inverse matrix is

$$A^{-1} = \begin{bmatrix} 1-10/(k-2) & 5/(k-2) & -2 \\ 6/(k-2) & -3/(k-2) & 1 \\ -2/(k-2) & 1/(k-2) & 0 \end{bmatrix}$$

(5i) $\det A = 2$, $\det B = -4$.

(5ii) $\det(A^{25}) = \det(A \cdot A \cdots A) = (\det A)(\det A)\dots(\det A) = (\det A)^{25} = 2^{25}$.

(5iii) There's no simple rule for $\det(A+B)$ so we first compute $A+B$:

$$\det(A+B) = \begin{vmatrix} 1+2 & 3+4 \\ 0+1 & 2+0 \end{vmatrix} = \begin{vmatrix} 3 & 7 \\ 1 & 2 \end{vmatrix} = -1.$$

(5iv) $\det(A^2BAB^2) = (\det A)^2(\det B)(\det A)(\det B^2) = 2^2 \times 4 \times 2 \times 4^2 = \dots$.

(5v) $\det(-C) = (-1)^2 \det C = \det C = 2$, and $\det(C^{-1}) = 1/\det C = 1/2$.