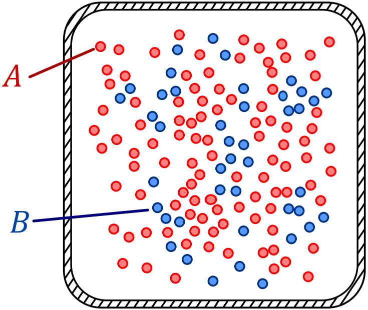
Euler's method for solving a differential equation (approximately)

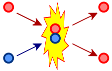
Math 320

Department of Mathematics, UW - Madison

February 28, 2011

A chemical reaction





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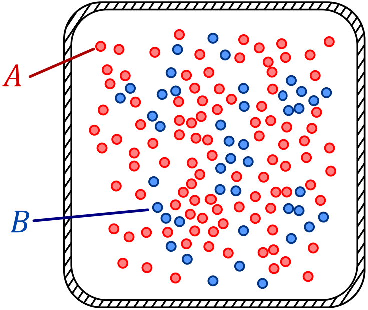

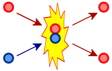

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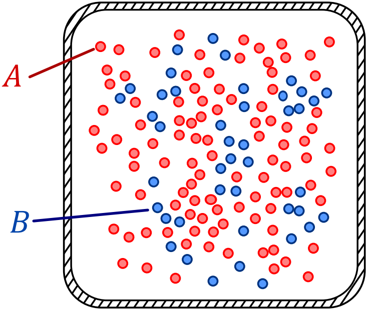

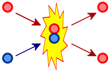

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Whenever an A and B molecule bump into each other the B turns into an A:

$$A + B \longrightarrow 2A$$

As the reaction proceeds, all B gets converted to A. How long does this take?

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Every time a reaction takes place, the ratio $x(t)$ increases, so

$$\frac{dx}{dt} \text{ is proportional to the reaction rate.}$$

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$$\frac{dx}{dt} = x(1-x).$$

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The solution is

$$x(t) = \frac{1}{1+49e^{-t}}.$$

The calculus solution (1)

You learned how to solve this diffeq in calculus. The differential equation

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\implies & \int \frac{dx}{x(1-x)} = \int dt \\
\implies & \ln|x| - \ln|1-x| + C_1 = t + C_2 \\
\implies & \ln\left|\frac{x}{1-x}\right| = t + C
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If you solve this for x you get the general solution. We can simplify our lives by being less ambitious and only look for solutions which satisfy $0 < x < 1$. This assumption allows us to drop the absolute value signs.

The calculus solution (2)

Solving for x we get

$$\ln \frac{x}{1-x} = t + C \implies x = \frac{e^{t+C}}{e^{t+C}+1} = \frac{1}{1+e^{-t-C}}. \qquad (1)$$

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This implies that the solution we are looking for is

$$x = \frac{1}{1+e^{-t-C}} = \frac{1}{1+49e^{-t}}.$$

Leonhard "$e^{\pi i} + 1 = 0$" Euler (1707 - 1783)

Solving $\frac{dx}{dt} = x(1-x)$

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I can't solve the equation because I don't know what $\frac{dx}{dt}$ is. So pick a small number $h > 0$ and say that

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Euler's (approximate) solution

Pick a small number $h > 0$, and compute

$$x(h) \quad = \quad x(0) + h \cdot x(0)[1 - x(0)]$$

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Now let's choose $h = 0.2$ and $x(0) = 0.02$, and compute $x(0.2)$, $x(0.4)$, $x(0.6)$, $x(0.8)$, $x(1.0)$, ...

Doing the calculations

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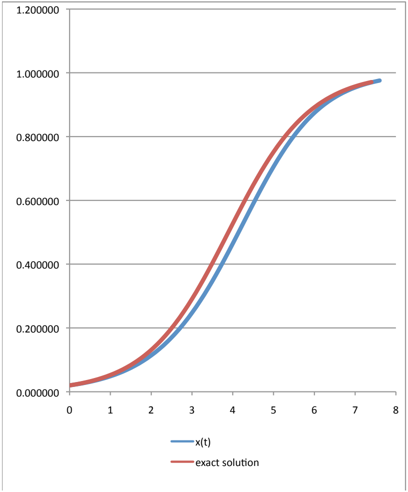
How do we do this in the 21st century? With a computer.

For more complicated diffeqs one should learn to program a computer, but for the example we've been looking at you can get Excel (or some other spreadsheet program like Open Office) to compute and plot the solutions.

What the spreadsheet computed

Here are the numbers, and graphs. The exact solution is $x(t) = 1/(1 + 49e^{-t})$.

h	t	x(t)	x'(t)	exact solution
0.2	0	0.020000	0.019600	0.020000
0.2	0.2	0.023920	0.023348	0.024320
0.2	0.4	0.028590	0.027772	0.029546
0.2	0.6	0.034144	0.032978	0.035853
0.2	0.8	0.040740	0.039080	0.043446
0.2	1	0.048556	0.046198	0.052559
0.2	1.2	0.057795	0.054455	0.063458
0.2	1.4	0.068686	0.063968	0.076434
0.2	1.6	0.081480	0.074841	0.091803
0.2	1.8	0.096448	0.087146	0.109894
0.2	2	0.113877	0.100909	0.131037
0.2	2.2	0.134059	0.116087	0.155537
0.2	2.4	0.157277	0.132541	0.183649
0.2	2.6	0.183785	0.150008	0.215545
0.2	2.8	0.213786	0.168082	0.251276
0.2	3	0.247403	0.186195	0.290734
0.2	3.2	0.284642	0.203621	0.333628
0.2	3.4	0.325366	0.219503	0.379465
0.2	3.6	0.369266	0.232909	0.427558
0.2	3.8	0.415848	0.242918	0.477061
0.2	4	0.464432	0.248735	0.527019
0.2	4.2	0.514179	0.249799	0.576441
0.2	4.4	0.564138	0.245886	0.624380
0.2	4.6	0.613316	0.237160	0.669999
0.2	4.8	0.660748	0.224160	0.712628
0.2	5	0.705580	0.207737	0.751790
0.2	5.2	0.747127	0.188928	0.787208
0.2	5.4	0.784913	0.168825	0.818791
0.2	5.6	0.818678	0.148445	0.846600
0.2	5.8	0.848367	0.128641	0.870815
0.2	6	0.874095	0.110053	0.891696
0.2	6.2	0.896105	0.093101	0.909552
0.2	6.4	0.914725	0.078003	0.924713
0.2	6.6	0.930326	0.064820	0.937508
0.2	6.8	0.943290	0.053494	0.948249
0.2	7	0.953989	0.043894	0.957229
0.2	7.2	0.962768	0.035846	0.964708
0.2	7.4	0.969937	0.029159	0.970920
0.2	7.6	0.975769	0.023644	



Point and click on-line diffeq solver

There are several graphical on-line solvers for differential equations. If you go to this web page:

http://virtualmathmuseum.org/ODE/1o1d-MassAction

you can see graphs of the solution to our equation $\frac{dx}{dt} = x(1 - x)$.