

INHOMOGENEOUS LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

1. DEFINITIONS AND A GENERAL FACT

If A is an $n \times n$ matrix and $\mathbf{f}(t)$ is some given vector function, then the system of differential equations

$$(1) \quad \mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{f}(t)$$

is said to be linear inhomogeneous. The given Right Hand Side $\mathbf{f}(t)$ is sometimes called the “forcing term.” The corresponding homogeneous equation is the same equation in which $\mathbf{f}(t)$ has been replaced by zero:

$$(2) \quad \mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}.$$

1.1. Theorem. *The general solution of the inhomogeneous system of equations (1) is*

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t),$$

where $\mathbf{x}_h(t)$ is the general solution to the homogeneous equation, and $\mathbf{x}_p(t)$ is any **particular** solution to the inhomogeneous equation.

Since we know how to solve the homogeneous equation using the eigenvalues and eigenvectors of A , “all we still have to do” is find a particular solution.

2. FINDING PARTICULAR SOLUTIONS BY GUESSING

2.1. Exponential forcing terms. If the “forcing term” $\mathbf{f}(t)$ is of the form

$$\mathbf{f}(t) = e^{\alpha t} \mathbf{v}$$

where α is a constant, and $\mathbf{v}$ is a constant vector, then try to find a solution of the form

$$\mathbf{x}_p(t) = e^{\alpha t} \mathbf{w}$$

Substitution in the diffeq leads to

$$\alpha e^{\alpha t} \mathbf{w} - A(e^{\alpha t} \mathbf{w}) = e^{\alpha t} \mathbf{v}.$$

Cancelling exponentials then gives you the following equation for the vector $\mathbf{w}$

$$(3) \quad (\alpha I - A)\mathbf{w} = \mathbf{v}.$$

If α is not an eigenvalue of A then this system can be solved, and the solution is

$$\mathbf{w} = (\alpha I - A)^{-1} \mathbf{v}$$

so that the particular solution of the diffeq is

$$(4) \quad \mathbf{x}_p(t) = e^{\alpha t} (\alpha I - A)^{-1} \mathbf{v}$$

2.2. Trigonometric forcing terms. If the forcing term is

$$\mathbf{f}(t) = \cos(\omega t)\mathbf{a} + \sin(\omega t)\mathbf{b}$$

then most of the time a particular solution of the same form can be found.

You can substitute a particular solution of the form

$$\mathbf{x}_p(t) = \cos(\omega t)\mathbf{p} + \sin(\omega t)\mathbf{q}$$

and solve for the vectors $\mathbf{p}$ and $\mathbf{q}$. The resulting algebra becomes complicated and it is better to use a detour through complex numbers.

The trick to remember is that the forcing term

$$\mathbf{f}(t) = \cos(\omega t)\mathbf{a} + \sin(\omega t)\mathbf{b}$$

is the real part of

$$\cos(\omega t)\mathbf{a} + \sin(\omega t)\mathbf{b} + i\{\sin(\omega t)\mathbf{a} + \cos(\omega t)\mathbf{b}\},$$

which can be written more shortly as

$$(\mathbf{a} - i\mathbf{b})(\cos \omega t + i \sin \omega t) = \mathbf{c}e^{i\omega t}$$

if $\mathbf{c} = \mathbf{a} - i\mathbf{b}$.

So instead of solving the original equation $\mathbf{x}'_p - A\mathbf{x}_p = \mathbf{f}(t)$ you solve the complex version of this equation, namely

$$(5) \quad \mathbf{z}'_p - A\mathbf{z}_p = \mathbf{c}e^{i\omega t}.$$

Then the real part

$$\mathbf{x}_p(t) = \Re[\mathbf{z}_p(t)]$$

is a solution of the original equation $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$.

You find $\mathbf{z}_p(t)$ as in the previous case, namely, try

$$\mathbf{z}_p(t) = e^{i\omega t}\mathbf{d}$$

where $\mathbf{d}$ is an undetermined complex vector. Substitution yields

$$(i\omega I - A)\mathbf{d} = \mathbf{c} \implies \mathbf{d} = (i\omega I - A)^{-1}\mathbf{c}.$$

Thus $\mathbf{d}$ can be computed by solving the system of equations $(i\omega I - A)\mathbf{d} = \mathbf{c}$, e.g. by row reducing. The solution will be a some complex vector $\mathbf{d}$. Multiply with $e^{i\omega t}$, using Euler's formula, and take the real part to get the real particular solution $\mathbf{x}_p(t) = \Re[e^{i\omega t}\mathbf{d}]$.

3. VARIATION OF CONSTANTS

3.1. Description of the general method. Suppose you have already solved the homogeneous equation $\mathbf{x}' - A\mathbf{x} = 0$, and that your general solution is

$$(6) \quad \mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t).$$

If you now want to find the solution to the inhomogeneous equation

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{f}(t)$$

then there is a trick, called *variation of constants* or *variation of parameters* that will give you that solution.

The trick is to look for a solution of the form

$$(7) \quad \mathbf{x}(t) = c_1(t)\mathbf{x}_1(t) + \cdots + c_n(t)\mathbf{x}_n(t).$$

This is almost the same as the general solution to the homogeneous equation, except here the quantities $c_1, \dots, c_n$ are no longer constants: we allow them to vary, and this gives the method its name.

The method consists of substituting (7) in the inhomogeneous equation, and deriving equations for the “varying constants” $c_1(t), \dots, c_n(t)$. When you do the substitution, many terms will cancel, and you always end up with the following:

If $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are solutions of the homogeneous equation, and if $\mathbf{x}_p(t)$ is given by (7), then

$$(8) \quad \mathbf{x}'_p(t) - A\mathbf{x}_p(t) = c'_1(t)\mathbf{x}_1(t) + \dots + c'_n(t)\mathbf{x}_n(t).$$

Since we want the right hand side $\mathbf{x}'_p - A\mathbf{x}_p$ to equal the forcing term $\mathbf{f}(t)$, so we must solve the equation(s)

$$(9) \quad c'_1(t)\mathbf{x}_1(t) + \dots + c'_n(t)\mathbf{x}_n(t) = \mathbf{f}(t)$$

for the derivatives $c'_1(t), \dots, c'_n(t)$.

Once we have found $c'_1(t), \dots, c'_n(t)$ we can integrate to find the “varying constants” $c_1(t), \dots, c_n(t)$, and therefore also the solution $\mathbf{x}_p(t)$.

3.2. When you have n eigenvectors and eigenvalues. Typically you will have solved the homogeneous equation by computing the eigenvalues

$$\lambda_1, \dots, \lambda_n$$

and corresponding eigenvectors

$$\mathbf{v}_1, \dots, \mathbf{v}_n$$

of the matrix A (So $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, etc.). If this is the case then $\mathbf{x}_j(t) = e^{\lambda_j t}\mathbf{v}_j$ is a solution, and the general solution you found is

$$\mathbf{x}(t) = c_1 \underbrace{e^{\lambda_1 t}\mathbf{v}_1}_{\mathbf{x}_1(t)} + \dots + c_n \underbrace{e^{\lambda_n t}\mathbf{v}_n}_{\mathbf{x}_n(t)}.$$

In this case the method of Variation of Constants tells you to try a particular solution of the form

$$\mathbf{x}(t) = c_1(t) \underbrace{e^{\lambda_1 t}\mathbf{v}_1}_{\mathbf{x}_1(t)} + \dots + c_n(t) \underbrace{e^{\lambda_n t}\mathbf{v}_n}_{\mathbf{x}_n(t)}$$

where the constants c_i have been replaced by functions of t .

Substituting $\mathbf{x}(t)$ in the equation $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$ is made easy by using the handy formula (8) which, in the present case, says

$$(10) \quad \mathbf{x}' - A\mathbf{x} = c'_1(t)e^{\lambda_1 t}\mathbf{v}_1 + \dots + c'_n(t)e^{\lambda_n t}\mathbf{v}_n.$$

We now must set this equal to the forcing term $\mathbf{f}(t)$. To do this we write the forcing term as a linear combination of the eigenvectors

$$(11) \quad \mathbf{f}(t) = f_1(t)\mathbf{v}_1 + \dots + f_n(t)\mathbf{v}_n$$

Finding $f_1(t), \dots, f_n(t)$ requires you to solve a system of n linear equations, which you can do by row reduction.

The equation $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$ together with (10) and (11) then implies

$$(12) \quad c'_1(t)e^{\lambda_1 t} = f_1(t), \quad \dots \quad c'_n(t)e^{\lambda_n t} = f_n(t),$$

and thus

$$c_1(t) = \int e^{-\lambda_1 t} f_1(t) dt, \quad \dots \quad c_n(t) = \int e^{-\lambda_n t} f_n(t) dt.$$

Reinsert these formulas for $c_i(t)$ in the original definition of $\mathbf{x}(t)$ and you get

$$(13) \quad \mathbf{x}(t) = \left[\int e^{-\lambda_1 t} f_1(t) dt \right] e^{\lambda_1 t} \mathbf{v}_1 + \dots + \left[\int e^{-\lambda_n t} f_n(t) dt \right] e^{\lambda_n t} \mathbf{v}_n$$

The integrals between brackets $[\dots]$ are indefinite integrals, and therefore contain constants. So the solution $\mathbf{x}(t)$ in this formula has n constants in it, and it is actually the general solution to the inhomogeneous equation. If all you want is a formula for the general solution then you could stop here: (13) is your solution.

3.3. The particular solution with zero initial data. There is one particular solution which is of interest, namely, the solution $\mathbf{x}(t)$ which starts out with $\mathbf{x}(0) = \mathbf{0}$. Why? Suppose that you had to solve the initial value problem

$$\begin{cases} \mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{f}(t) \\ \mathbf{x}(0) = \mathbf{a} \end{cases}$$

where $\mathbf{a}$ is the given initial value. As before, we write the answer as

$$(14) \quad \mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t),$$

but now we also specify the initial values of both $\mathbf{x}_h(t)$ and $\mathbf{x}_p(t)$: let $\mathbf{x}_h$ be the solution of

$$(15) \quad \mathbf{x}_h'(t) - A\mathbf{x}_h(t) = \mathbf{0}, \quad \mathbf{x}_h(0) = \mathbf{a}$$

and let $\mathbf{x}_p(t)$ be the solution of

$$(16) \quad \mathbf{x}_p'(t) - A\mathbf{x}_p(t) = \mathbf{f}(t), \quad \mathbf{x}_p(0) = \mathbf{0}$$

If you think of the solution $\mathbf{x}(t)$ as the “response” to the “input” $\mathbf{f}(t)$, then equation (14) splits the response $\mathbf{x}(t)$ into a component $\mathbf{x}_h(t)$ which is due to the initial conditions, and another component $\mathbf{x}_p(t)$ which is due to the forcing term.

You can find $\mathbf{x}_h(t)$ by solving the homogeneous equation and choosing the constants c_i so that $\mathbf{x}_h(0) = \mathbf{a}$.

You can find the forcing term by using Variation of Constants as above, and requiring $c_1(0) = 0, \dots, c_n(0) = 0$. Then the equations (12) (and the Fundamental Theorem of Calculus)¹ imply

$$(17) \quad \begin{cases} c_1(t) = \int_0^t c_1'(s) ds = \int_0^t e^{-\lambda_1 s} f_1(s) ds \\ \vdots \\ c_n(t) = \int_0^t e^{-\lambda_n s} f_n(s) ds \end{cases}$$

Finally you substitute these integrals in the definition (21) of $\mathbf{x}_p(t)$, and you find

$$(18) \quad \mathbf{x}_p(t) = \underbrace{\left[\int_0^t e^{-\lambda_1 s} f_1(s) ds \right]}_{c_1(t)} e^{\lambda_1 t} \mathbf{v}_1 + \dots + \underbrace{\left[\int_0^t e^{-\lambda_n s} f_n(s) ds \right]}_{c_n(t)} e^{\lambda_n t} \mathbf{v}_n$$

¹The Fundamental Theorem of Calculus says $c(t) - c(0) = \int_0^t c'(s) ds$, so if $c(0) = 0$ then $c(t) = \int_0^t c'(s) ds$

One often rewrites the coefficients of the vectors $\mathbf{v}_i$ as follows:

$$(19) \quad \underbrace{\left[\int_0^t e^{-\lambda_1 s} f_1(s) ds \right]}_{c_1(t)} e^{\lambda_1 t} = \int_0^t e^{\lambda_1 t} e^{-\lambda_1 s} f_1(s) ds = \int_0^t e^{\lambda_1(t-s)} f_1(s) ds.$$

The particular solution with zero initial data is then

$$\mathbf{x}_p(t) = \left[\int_0^t e^{\lambda_1(t-s)} f_1(s) ds \right] \mathbf{v}_1 + \cdots + \left[\int_0^t e^{\lambda_n(t-s)} f_n(s) ds \right] \mathbf{v}_n.$$

3.4. Convolution integrals. Integrals of the form

$$\int_0^t g(t-s) f(s) ds$$

are called **convolution integrals**, and they show up very often when you solve constant coefficient linear differential equations.

The coefficient

$$\int_0^t e^{\lambda_1(t-s)} f_1(s) ds.$$

of $\mathbf{v}_1$ in the particular solution $\mathbf{x}_p$ is a convolution integral with $g(t) = e^{\lambda_1 t}$. It has the following interpretation: the $\mathbf{v}_1$ component of the response is given by Since $f_1(s)$ is the $\mathbf{v}_1$ component of the input at time s , the $\mathbf{v}_1$ component of the solution is an average of the $\mathbf{v}_1$ components $f_1(s)$ of the input at earlier times s . The weight with which $f_1(s)$ contributes to the solution at time t only depends on the time delay $t-s$.

4. AN EXAMPLE

Let $y(t)$ be the solution of

$$(20) \quad \begin{cases} y''(t) + 3y'(t) + 2y(t) = F(t), \\ y(0) = y'(0) = 0. \end{cases}$$

First we rewrite this as a system of first order equations: let $x_1 = y$, $x_2 = y'$. Then

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 3x_2 - F(t), \end{cases}$$

so $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfies

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t) \text{ with } A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \text{ and } \mathbf{f}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.$$

The eigenvalues and vectors of A are

$$\lambda_1 = -1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = -2, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Therefore we should find a particular solution of the form

$$(21) \quad \mathbf{x}_p(t) = c_1(t)e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2(t)e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The initial conditions are

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

We get these initial conditions if we require

$$(22) \quad c_1(0) = c_2(0) = 0$$

Substituting $\mathbf{x}_p(t)$ from (21) in $\mathbf{x}' - A\mathbf{x}$ leads to

$$(23) \quad \mathbf{x}'_p(t) - A\mathbf{x}_p(t) = c'_1(t)e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c'_2(t)e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

by the handy identity (10). To compare this with $\mathbf{f}(t)$ we must write the forcing term $\mathbf{f}(t)$ as a combination of the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{f}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix} = f_1(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + f_2(t) \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 \\ -f_1 - f_2 \end{pmatrix}.$$

Solve this by row reduction:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -2 & F(t) \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & F(t) \\ 0 & 1 & -F(t) \end{array} \right] \implies f_1(t) = F(t), f_2(t) = -F(t).$$

Our decomposition of the forcing term $\mathbf{f}(t)$ into the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ is therefore

$$(24) \quad \mathbf{f}(t) = F(t)\mathbf{v}_1 - F(t)\mathbf{v}_2$$

Now combining (24) and (23) we see that $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$ implies

$$c'_1(t)e^{-t}\mathbf{v}_1 + c'_2(t)e^{-2t}\mathbf{v}_2 = F(t)\mathbf{v}_1 - F(t)\mathbf{v}_2$$

so that

$$c'_1(t)e^{-t} = F(t) \text{ and } c'_2(t)e^{-2t} = -F(t).$$

Solving for c'_1 and c'_2 , and integrating, taking into account that $c_1(0) = c_2(0) = 0$, you get

$$c_1(t) = \int_0^t e^s F(s) ds \text{ and } c_2(t) = - \int_0^t e^{2s} F(s) ds.$$

Put this back into the solution $\mathbf{x}_p(t)$:

$$\mathbf{x}_p(t) = c_1(t)e^{-t}\mathbf{v}_1 + c_2(t)e^{-2t}\mathbf{v}_2 = \left[\int_0^t e^{s-t} F(s) ds \right] \mathbf{v}_1 - \left[\int_0^t e^{2(s-t)} F(s) ds \right] \mathbf{v}_2$$

To see the solution in all its components remember that $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, so that

$$\mathbf{x}_p(t) = \left[\int_0^t e^{s-t} F(s) ds \right] \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \left[\int_0^t e^{2(s-t)} F(s) ds \right] \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Adding these vectors produces a large formula. Since $\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$ the top component of $\mathbf{x}_p(t)$ is the solution $y(t)$ to the second order equation we began with. It is given by

$$y(t) = \int_0^t e^{s-t} F(s) ds - \int_0^t e^{2(s-t)} F(s) ds = \int_0^t \{e^{-(t-s)} - e^{-2(t-s)}\} F(s) ds.$$

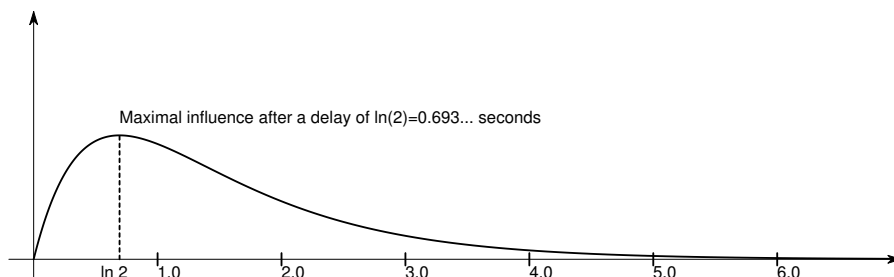
This integral is of the form

$$\int_0^t K(t-s)F(s) ds$$

where

$$K(\tau) = e^{-\tau} - e^{-2\tau}.$$

The interpretation of the convolution integral is that the solution $y(t)$ is a response to all the inputs $F(s)$ for $0 < s < t$, and that the contribution of the input $F(s)$ at time s is weighed by a factor $K(t - s)$.



A graph of $K(\tau)$; $K(\tau)$ is maximal at $\tau = \ln 2$

5. PROBLEMS

1. Use the method of variation of Constants to find the solution to $y'' + 5y' + 6y = F(t)$, with $y(0) = y'(0) = 0$. More specifically:
 - (i) rewrite the equation as a system of first order differential equations of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$; what are A and $\mathbf{f}(t)$ in this problem?
 - (ii) Find the eigenvalues and eigenvectors of A .
 - (iii) Find the general solution to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$.
 - (iv) Which kind of solution do you try when you use V.o.C. to find a particular solution?
 - (v) which initial conditions should you impose on $\mathbf{x}$ if you want $y(0) = y'(0) = 0$?
 - (vi) Find the solution $\mathbf{x}(t)$.
 - (vii) Find the solution $y(t)$.
2. Do the same for $y'' - y = F(t)$, with $y(0) = y'(0) = 0$.
3. Use V.o.C. to find the solution to

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t), \text{ with } A = \begin{pmatrix} -10 & 0 & 0 \\ 10 & -5 & 0 \\ 0 & 5 & -4 \end{pmatrix}, \text{ and } \mathbf{f}(t) = \begin{pmatrix} F(t) \\ 0 \\ 0 \end{pmatrix}.$$

Write a formula for $x_3(t)$ (the third component of $\mathbf{x}(t)$): the answer will be a convolution integral of the form

$$\int_0^t K(t-s)F(s) ds$$

Identify $K(t-s)$.

(There is an interpretation of this problem as in example 2 on page 496: in the present problem the inflow of brine into the first tank is given by $F(t)$, rather than just “20” as in the book.)

6. PROOF OF FORMULA (8)

The chain rule tells us that

$$\begin{aligned} \mathbf{x}'(t) &= c_1(t)\mathbf{x}'_1(t) + \cdots + c_n(t)\mathbf{x}'_n(t) \\ &\quad + c'_1(t)\mathbf{x}_1(t) + \cdots + c'_n(t)\mathbf{x}_n(t). \end{aligned}$$

Since the $\mathbf{x}_j(t)$ are solutions to the homogeneous equation, we get

$$\begin{aligned}\mathbf{x}'(t) &= c_1(t)A\mathbf{x}_1(t) + \cdots + c_n(t)A\mathbf{x}_n(t) \\ &\quad + c'_1(t)\mathbf{x}_1(t) + \cdots + c'_n(t)\mathbf{x}_n(t).\end{aligned}$$

On the other hand we also have

$$A\mathbf{x}(t) = c_1(t)A\mathbf{x}_1(t) + \cdots + c_n(t)A\mathbf{x}_n(t),$$

and therefore

$$(25) \quad \mathbf{x}'(t) - A\mathbf{x}(t) = c'_1(t)\mathbf{x}_1(t) + \cdots + c'_n(t)\mathbf{x}_n(t).$$