PRACTICE FINAL FOR MATH 320, SPRING 2010

Instructions: For every problem give a short outline of your approach before you compute anything. Then solve the problem.

1. Find the general solution to

$$\mathbf{x}''(t) + A\,\mathbf{x} = 0, \text{ where } A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

2. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 3 & 2 \\ -2 & -3 & 0 & 4 \end{pmatrix}$$

3. Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 4 & 2 \\ -1 & 2 & 4 \end{pmatrix}$$

(or any of the other matrices in problems 1...32 of §6.1, p.374-375)

4. The eigenvalues and vectors of the matrix A are

$$\lambda_1 = -4,\ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \text{ and } \lambda_2 = -6,\ \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(***i***) Use the method of variation of constants to solve

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \begin{pmatrix} F(t) \\ 0 \end{pmatrix}, \text{ with } \mathbf{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If you found the right solution, then the second component $x_2(t)$ of $\mathbf{x}(t)$ can be written in the form

$$x_2(t) = \int_0^t K(t-s)F(s)\,ds$$

(***ii***) Find $K(t-s)$.

5. Find the general solution to the inhomogeneous system of linear differential equations

$$\mathbf{x}' = A\mathbf{x} + e^{i\omega t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ where } A = \begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix} \text{ and } \omega \text{ is arbitrary.}$$

Your solution should be valid for ***almost*** all possible values of the forcing frequency ω: if there are one or two values for which your solution is not valid, indicate those values.

ANSWERS

(1) First: read the book, page 434.

Try solutions of the form $\mathbf{x}(t) = e^{\alpha t}\mathbf{v}$, and you find

$$\alpha^2 e^{\alpha t}\mathbf{v} + e^{\alpha t}A\mathbf{v} = \mathbf{0} \implies A\mathbf{v} = -\alpha^2\mathbf{v}.$$

So $\mathbf{v}$ must be an eigenvalue and $-\alpha^2$ must be the corresponding eigenvalue.

Now look for eigenvectors/values of A. You find

$$\lambda_1 = 6,\ \mathbf{v}_1 = \begin{pmatrix}1\\1\end{pmatrix} \text{ and } \lambda_2 = 2,\ \mathbf{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}.$$

We therefore get solutions for $-\alpha^2 = 6$ and for $-\alpha^2 = 2$. In the case $-\alpha^2 = 6$ there are two $-i\alpha$'s with $\alpha^2 = 6$ so we get ***two*** solutions, $\alpha = \pm\sqrt{6}$:

$$e^{i\sqrt{6}t}\begin{pmatrix}1\\1\end{pmatrix}, \text{ and } e^{-i\sqrt{6}t}\begin{pmatrix}1\\1\end{pmatrix}.$$

For $-\alpha^2 = 2$ we get another two solutions,

$$e^{i\sqrt{2}t}\begin{pmatrix}1\\-1\end{pmatrix}, \text{ and } e^{-i\sqrt{2}t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

The general solution is therefore

$$\mathbf{x}(t) = c_1 e^{i\sqrt{6}t}\begin{pmatrix}1\\1\end{pmatrix} + c_2 e^{-i\sqrt{6}t}\begin{pmatrix}1\\1\end{pmatrix} + c_3 e^{i\sqrt{2}t}\begin{pmatrix}1\\-1\end{pmatrix} + c_4 e^{-i\sqrt{2}t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

The real form of the general solution is

$$a_1\cos\sqrt{6}t\begin{pmatrix}1\\1\end{pmatrix} + a_2\sin\sqrt{6}t\begin{pmatrix}1\\1\end{pmatrix} + a_3\cos\sqrt{2}t\begin{pmatrix}1\\-1\end{pmatrix} + a_4\sin\sqrt{2}t\begin{pmatrix}1\\-1\end{pmatrix}.$$

(2) They are $1, 2, 3, 4$.

(3) They are

$$\lambda_1 = -1, \mathbf{v}_1 = \begin{pmatrix}3\\-1\\1\end{pmatrix}, \quad \lambda_2 = 6, \mathbf{v}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \quad \lambda_3 = 2, \mathbf{v}_3 = \begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

(4i) The solution to the homogeneous equation is

$$\mathbf{x}_h(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

This satisfies $\mathbf{x}_h' - A\mathbf{x}_h = \mathbf{0}$. To find a function which satisfies $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$, we try

$$\mathbf{x}(t) = c_1(t) e^{\lambda_1 t}\mathbf{v}_1 + c_2(t) e^{\lambda_2 t}\mathbf{v}_2.$$

Before we compute anything, let's look at the initial values. At $t = 0$ we are given

$$\mathbf{x}(0) = \mathbf{0} \text{ and also } \mathbf{x}(0) = c_1(0)\mathbf{v}_1 + c_2(0)\mathbf{v}_2.$$

So we must choose

$$c_1(0) = 0 \text{ and } c_2(0) = 0.$$

Set this aside, we'll need it later on.

Now we substitute $\mathbf{x}(t)$ in $\mathbf{x}' - A\mathbf{x}$. If you blindly do that you get a longish calculation, but if you have read the handout about Variation of Constants then you know that the result always is

$$\mathbf{x}' - A\mathbf{x} = c_1'(t) e^{\lambda_1 t}\mathbf{v}_1 + c_2'(t) e^{\lambda_2 t}\mathbf{v}_2.$$

We want $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$, so we have to solve

$$c_1'(t) e^{\lambda_1 t}\mathbf{v}_1 + c_2'(t) e^{\lambda_2 t}\mathbf{v}_2 = \mathbf{f}(t)$$

for $c_1'(t)e^{\lambda_1 t}$ and $c_2'(t)e^{\lambda_2 t}$. This is the right moment to remember what $\lambda_{\dots}$ and $\mathbf{v}_{\dots}$ are. We have to solve

$$c_1'(t) e^{-4t}\begin{pmatrix}1\\2\end{pmatrix} + c_2'(t) e^{-6t}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}.F(t)\\0\end{pmatrix}.$$

for $c_1'(t)e^{-4t}$ and $c_2'(t)e^{-6t}$, which we do by row reducing

$$\left[\begin{array}{cc|c} 1 & 2 & F(t) \\ 2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{3}F(t) \\ 0 & 1 & \frac{2}{3}F(t) \end{array}\right]$$

So

$$c_1'(t) = -\frac{1}{3}e^{4t}F(t) \text{ and } c_2'(t) = \frac{2}{3}e^{6t}F(t).$$

The fundamental theorem of calculus says that $\int_a^b f'(x)dx = f(b) - f(a)$ for any function $f(x)$, so

$$c_1(t) - c_1(0) = \int_0^t c_1'(s)ds.$$

Now remember the initial values $c_1(0) = c_2(0) = 0$ which we had found in the beginning. This tells us that

$$c_1(t) = -\frac{1}{3}\int_0^t e^{4s}F(s)\,ds, \qquad c_2(t) = \frac{2}{3}\int_0^t e^{6s}F(s)\,ds.$$

Substitute this into our original definition of $\mathbf{x}(t)$ to get

$$\mathbf{x}(t) = e^{-4t}\int_0^t -\tfrac{1}{3}e^{4s}F(s)\,ds\begin{pmatrix}1\\2\end{pmatrix} + e^{-6t}\int_0^t \tfrac{2}{3}e^{6s}F(s)\,ds\begin{pmatrix}2\\1\end{pmatrix}$$

This is the solution. You can clean it up a bit by writing it as

$$\mathbf{x}(t) = \int_0^t -\tfrac{1}{3}e^{-4(t-s)}F(s)\,ds\begin{pmatrix}1\\2\end{pmatrix} + \int_0^t \tfrac{2}{3}e^{-6(t-s)}F(s)\,ds\begin{pmatrix}2\\1\end{pmatrix}$$

(4ii) The second component is

$$x_2(t) = \int_0^t \tfrac{2}{3}\left\{e^{-6(t-s)} - e^{-4(t-s)}\right\} F(s)\, ds$$

so

$$K(t-s) = \tfrac{2}{3}\left\{e^{-6(t-s)} - e^{-4(t-s)}\right\}$$

(5) The general solution to the equation is always

$$\mathbf{x}_{\text{gen}}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)$$

where $\mathbf{x}_p$ is some particular solution and $\mathbf{x}_h$ is the general solution to the homogeneous equation.

The homogeneous equation. Find the eigenvalues/vectors for A. They are

$$\lambda = 4i, \mathbf{v} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}, \qquad \bar{\lambda} = -4i, \bar{\mathbf{v}} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

Therefore the general solution to the homogeneous equation is

$$\mathbf{x}_h(t) = c_1 e^{4it} \begin{pmatrix} 1 \\ -2i \end{pmatrix} + c_2 e^{-4it} \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

This is the complex form of the general solution. To get the real form you take the real and imaginary parts of one of the complex solutions:

$$\begin{aligned} e^{4it} \begin{pmatrix} 1 \\ -2i \end{pmatrix} &= (\cos 4t + i \sin 4t) \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right\} \\ &= \cos 4t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 4t \begin{pmatrix} 0 \\ -2 \end{pmatrix} + i \left\{ \cos 4t \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \sin 4t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

The real and imaginary parts of a solution to a homogeneous equation are also solutions, so

$$\mathbf{x}_1(t) = \cos 4t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 4t \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

and

$$\mathbf{x}_2(t) = \cos 4t \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \sin 4t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

are solutions to $\mathbf{x}' - A\mathbf{x} = 0$. The general solution in real form is

$$\mathbf{x}_h(t) = a_1 \mathbf{x}_1(t) + a_2 \mathbf{x}_2(t).$$

The inhomogeneous equation. We look for a particular solution of the form $\mathbf{x}(t) = e^{i\omega t} \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$, and try to find the constants a and b. Substitute in the equation:

$$\mathbf{x}' - A\mathbf{x} = i\omega e^{i\omega t} \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) - e^{i\omega t} A \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \stackrel{?}{=} e^{i\omega t} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$$

Cancelling exponentials leads to the equations

$$\big(i\omega I - A\big)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1}$$

which we can solve for a and b by row reducing

$$\left[\begin{array}{cc|c} i\omega & 2 & 1 \\ -8 & i\omega & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 2-\frac{1}{8}\omega^2 & 1 \\ 1 & -\frac{1}{8}i\omega & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & \frac{1}{2-\frac{1}{8}\omega^2} \\ 1 & -\frac{1}{8}i\omega & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 1 & \frac{8}{16-\omega^2} \\ 1 & 0 & \frac{i\omega}{16-\omega^2} \end{array}\right]$$

If you don't like row reducing with complex numbers, then remember that all this is short hand for solving the equations (1), i.e.

$$\begin{aligned} i\omega a + 2b &= 1 \\ -8a + i\omega b &= 0 \end{aligned}$$

for a and b. Either way, the solution is

$$a = \frac{i\omega}{16-\omega^2}, \qquad b = \frac{8}{16-\omega^2}.$$

Therefore the particular solution is

$$\mathbf{x}_p(t) = e^{i\omega t}\begin{pmatrix} \frac{i\omega}{16-\omega^2} \\ \frac{8}{16-\omega^2} \end{pmatrix} = \frac{e^{i\omega t}}{16-\omega^2}\begin{pmatrix} i\omega \\ 8 \end{pmatrix}.$$

This solution is valid as long as $\omega^2 \neq 16$ (i.e. as long as we don't divide by zero), so as long as $\omega \neq \pm 4$.

The general solution. As we said above, the general solution to the inhomogeneous equation is $\mathbf{x}_{\text{gen}}(t) = \mathbf{x}_h(t) = \mathbf{x}_p(t)$, where $\mathbf{x}_h$ and $\mathbf{x}_p$ are the solutions we have just computed.