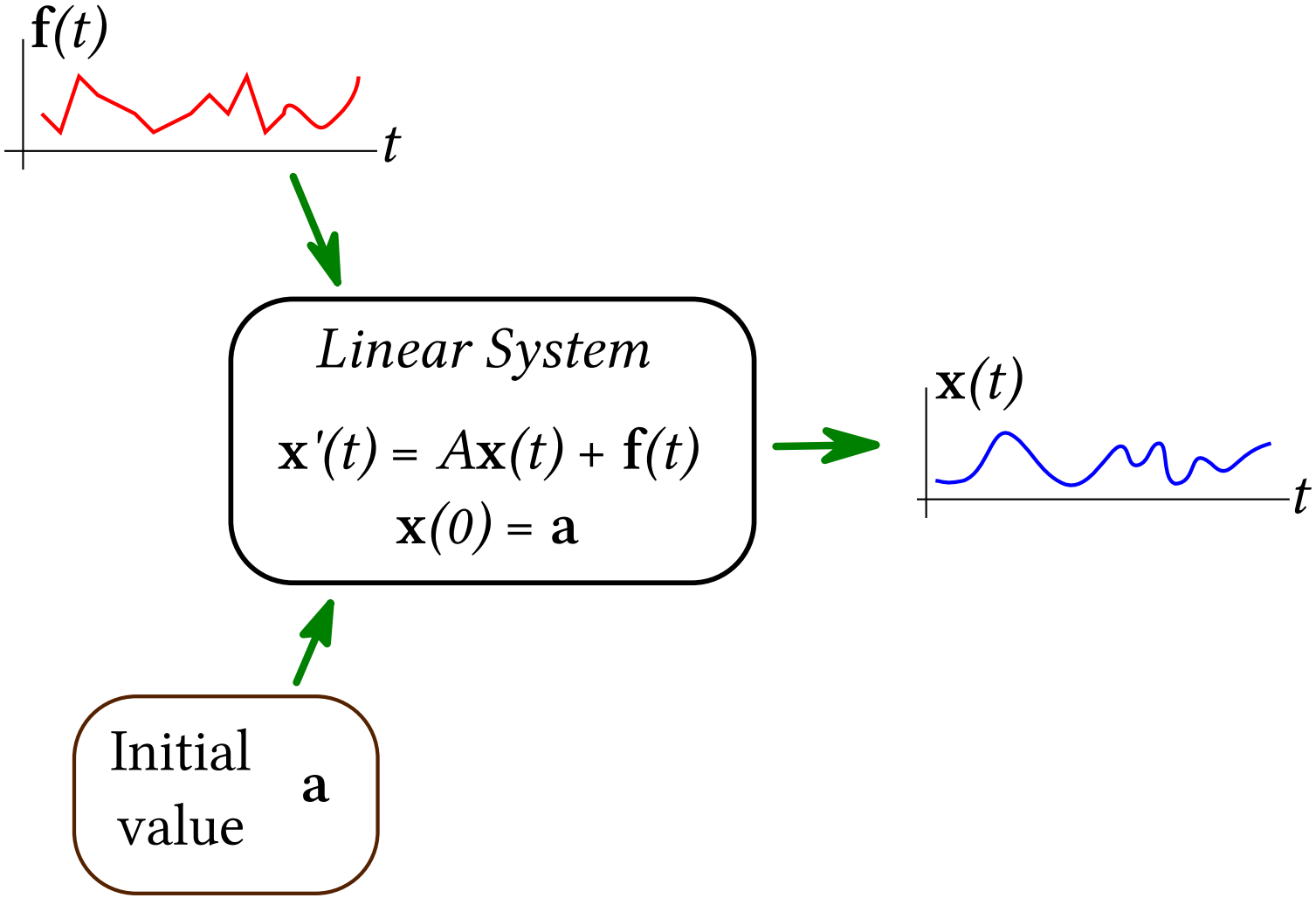
Math 320

Inhomogeneous linear systems of Differential Equations

May 7, 2010


f(t)
t
Linear System
x′(t) = Ax(t) + f(t)
x(0) = a
x(t)
t
Initial value
a

1 Definitions and a general fact

If A is an $n \times n$ matrix and $\mathbf{f}(t)$ is some given vector function, then the system of differential equations

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{f}(t) \tag{1}$$

is said to be linear inhomogeneous. The given Right Hand Side $\mathbf{f}(t)$ is sometimes called the "forcing term." The corresponding homogeneous equation is the same equation in which $\mathbf{f}(t)$ has been replaced by zero:

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}. \tag{2}$$

1.1 Theorem

The general solution of the inhomogeneous system of equations (1) *is*

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t),$$

where $\mathbf{x}_h(t)$ *is the general solution to the homogeneous equation, and* $\mathbf{x}_p(t)$ *is any* ***particular*** *solution to the inhomogeneous equation.*

Since we know how to solve the homogeneous equation using the eigenvalues and eigenvectors of A, "all we still have to do" is find a particular solution.

2 Finding Particular Solutions by Guessing

2.1 Exponential forcing terms

If the "forcing term" $\mathbf{f}(t)$ is of the form

$$\mathbf{f}(t) = e^{\alpha t}\mathbf{v}$$

where α is a constant, and $\mathbf{v}$ is a constant vector, then try to find a solution of the form

$$\mathbf{x}_p(t) = e^{\alpha t}\mathbf{w}$$

Substitution in the diffeq leads to

$$\alpha e^{\alpha t}\mathbf{w} - A\big(e^{\alpha t}\mathbf{w}\big) = e^{\alpha t}\mathbf{v}.$$

Cancelling exponentials then gives you the following equation for the vector $\mathbf{w}$

$$(\alpha I - A)\mathbf{w} = \mathbf{v}. \tag{3}$$

If α is not an eigenvalue of A then this system can be solved, and the solution is

$$\mathbf{w} = \big(\alpha I - A\big)^{-1}\mathbf{v}$$

so that the particular solution of the diffeq is

$$\mathbf{x}_p(t) = e^{\alpha t}\big(\alpha I - A\big)^{-1}\mathbf{v} \tag{4}$$

2.2 Trigonometric forcing terms

If the forcing term is

$$\mathbf{f}(t) = \cos(\omega t)\mathbf{a} + \sin(\omega t)\mathbf{b}$$

then most of the time a particular solution of the same form can be found.

You can substitute a particular solution of the form

$$\mathbf{x}_p(t) = \cos(\omega t)\mathbf{p} + \sin(\omega t)\mathbf{q}$$

and solve for the vectors $\mathbf{p}$ and $\mathbf{q}$. The resulting algebra becomes complicated and it is better to use a detour through complex numbers.

The trick to remember is that the forcing term

$$\mathbf{f}(t) = \cos(\omega t)\mathbf{a} + \sin(\omega t)\mathbf{b}$$

is the real part of

$$\cos(\omega t)\mathbf{a} + \sin(\omega t)\mathbf{b} + i\big\{\sin(\omega t)\mathbf{a} - \cos(\omega t)\mathbf{b}\big\},$$

which can be written more shortly as

$$(\mathbf{a} - i\mathbf{b})(\cos\omega t + i\sin\omega t) = \mathbf{c}e^{i\omega t}$$

if $\mathbf{c} = \mathbf{a} - i\mathbf{b}$.

So instead of solving the original equation $\mathbf{x}_p' - A\mathbf{x}_p = \mathbf{f}(t)$ you solve the complex version of this equation, namely

$$\mathbf{z}_p' - A\mathbf{z}_p = \mathbf{c}e^{i\omega t}. \tag{5}$$

Then the real part

$$\mathbf{x}_p(t) = \mathfrak{Re}\big[\mathbf{z}_p(t)\big]$$

is a solution of the original equation $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$.

You find $\mathbf{z}_p(t)$ as in the previous case, namely, try

$$\mathbf{z}_p(t) = e^{i\omega t}\mathbf{d}$$

where $\mathbf{d}$ is an undetermined complex vector. Substitution yields

$$\big(i\omega I - A\big)\mathbf{d} = \mathbf{c} \implies \mathbf{d} = \big(i\omega I - A\big)^{-1}\mathbf{c}.$$

Thus $\mathbf{d}$ can be computed by solving the system of equations $\big(i\omega I - A\big)\mathbf{d} = \mathbf{c}$, e.g. by row reducing. The solution will be a some complex vector $\mathbf{d}$. Multiply with $e^{i\omega t}$, using Euler's formula, and take the real part to get the real particular solution $\mathbf{x}_p(t) = \mathfrak{Re}[e^{i\omega t}\mathbf{d}]$.

3 Variation of Constants

3.1 Description of the general method

Suppose you have already solved the homogeneous equation $\mathbf{x}' - A\mathbf{x} = 0$, and that your general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t). \tag{6}$$

If you now want to find the solution to the inhomogeneous equation

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{f}(t)$$

then there is a trick, called ***variation of constants*** or ***variation of parameters*** that will give you that solution.

The trick is to look for a solution of the form

$$\mathbf{x}(t) = c_1(t)\mathbf{x}_1(t) + \cdots + c_n(t)\mathbf{x}_n(t). \tag{7}$$

This is almost the same as the general solution to the homogeneous equation, except here the quantities $c_1, \ldots, c_n$ are no longer constants: we allow them to vary, and this gives the method its name.

The method consists of substituting (7) in the inhomogeneous equation, and deriving equations for the "varying constants" $c_1(t), \ldots, c_n(t)$. When you do the substitution, many terms will cancel, and you always end up with the following:

If $\mathbf{x}_1(t)$***, . . . ,*** $\mathbf{x}_n(t)$ ***are solutions of the homogeneous equation, and if*** $\mathbf{x}_p(t)$ ***is given by***

$$\mathbf{x}_p(t) = c_1(t)\mathbf{x}_1(t) + \cdots c_n(t)\mathbf{x}_n(t) \tag{7}$$

then

$$\mathbf{x}_p'(t) - A\mathbf{x}_p(t) = c_1'(t)\mathbf{x}_1(t) + \cdots + c_n'(t)\mathbf{x}_n(t). \tag{8}$$

Since we want the right hand side $\mathbf{x}_p' - A\mathbf{x}_p$ to equal the forcing term $\mathbf{f}(t)$, so we must solve the equation(s)

$$c_1'(t)\mathbf{x}_1(t) + \cdots + c_n'(t)\mathbf{x}_n(t) = \mathbf{f}(t) \tag{9}$$

for the derivatives $c_1'(t), \ldots, c_n'(t)$.

Once we have found $c_1'(t), \ldots, c_n'(t)$ we can integrate to find the "varying constants" $c_1(t), \ldots, c_n(t)$, and therefore also the solution $\mathbf{x}_p(t)$.

3.2 When you have n eigenvectors and eigenvalues

Typically you will have solved the homogeneous equation by computing the eigenvalues

$$\lambda_1, \ldots, \lambda_n$$

and corresponding eigenvectors

$$\mathbf{v}_1, \ldots, \mathbf{v}_n$$

of the matrix A (So $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, etc.). If this is the case then $\mathbf{x}_j(t) = e^{\lambda_j t}\mathbf{v}_j$ is a solution, and the general solution you found is

$$\mathbf{x}(t) = c_1 \underbrace{e^{\lambda_1 t}\mathbf{v}_1}_{\mathbf{x}_1(t)} + \cdots + c_n \underbrace{e^{\lambda_n t}\mathbf{v}_n}_{\mathbf{x}_n(t)}.$$

In this case the method of Variation of Constants tells you to try a particular solution of the form

$$\mathbf{x}(t) = c_1(t) \underbrace{e^{\lambda_1 t}\mathbf{v}_1}_{\mathbf{x}_1(t)} + \cdots + c_n(t) \underbrace{e^{\lambda_n t}\mathbf{v}_n}_{\mathbf{x}_n(t)}$$

where the constants c_i have been replaced by functions of t.

Substituting $\mathbf{x}(t)$ in the equation $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$ is made easy by using the handy formula (8) which, in the present case, says

$$\mathbf{x}' - A\mathbf{x} = c_1'(t)e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n'(t)e^{\lambda_n t}\mathbf{v}_n. \tag{10}$$

We now must set this equal to the forcing term $\mathbf{f}(t)$. To do this we write the forcing term as a linear combination of the eigenvectors

$$\mathbf{f}(t) = f_1(t)\mathbf{v}_1 + \cdots + f_n(t)\mathbf{v}_n \tag{11}$$

Finding $f_1(t)$, ..., $f_n(t)$ requires you to solve a system of n linear equations, which you can do by row reduction.

The equation $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$ together with (10) and (11) then implies

$$c_1'(t)e^{\lambda_1 t} = f_1(t), \quad \ldots \quad c_n'(t)e^{\lambda_n t} = f_n(t), \tag{12}$$

and thus

$$c_1(t) = \int e^{-\lambda_1 t} f_1(t)\,dt, \quad \ldots \quad c_n(t) = \int e^{-\lambda_n t} f_n(t)\,dt.$$

Reinsert these formulas for $c_i(t)$ in the original definition of $\mathbf{x}(t)$ and you get

$$\mathbf{x}(t) = \left[\int e^{-\lambda_1 t} f_1(t)\,dt\right] e^{\lambda_1 t}\mathbf{v}_1 + \cdots + \left[\int e^{-\lambda_n t} f_n(t)\,dt\right] e^{\lambda_n t}\mathbf{v}_n \tag{13}$$

The integrals between brackets $[\cdots]$ are indefinite integrals, and therefore contain constants. So the solution $\mathbf{x}(t)$ in this formula has n constants in it, and it is actually the general solution to the inhomogeneous equation. If all you want is a formula for the general solution then you could stop here: (13) is your solution.

3.3 The particular solution with zero initial data

There is one particular solution which is of interest, namely, the solution $\mathbf{x}(t)$ which starts out with $\mathbf{x}(0) = \mathbf{0}$. Why? Suppose that you had to solve the initial value problem

$$\begin{cases} \mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{f}(t) \\ \mathbf{x}(0) = \mathbf{a} \end{cases}$$

where $\mathbf{a}$ is the given initial value. As before, we write the answer as

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t), \tag{14}$$

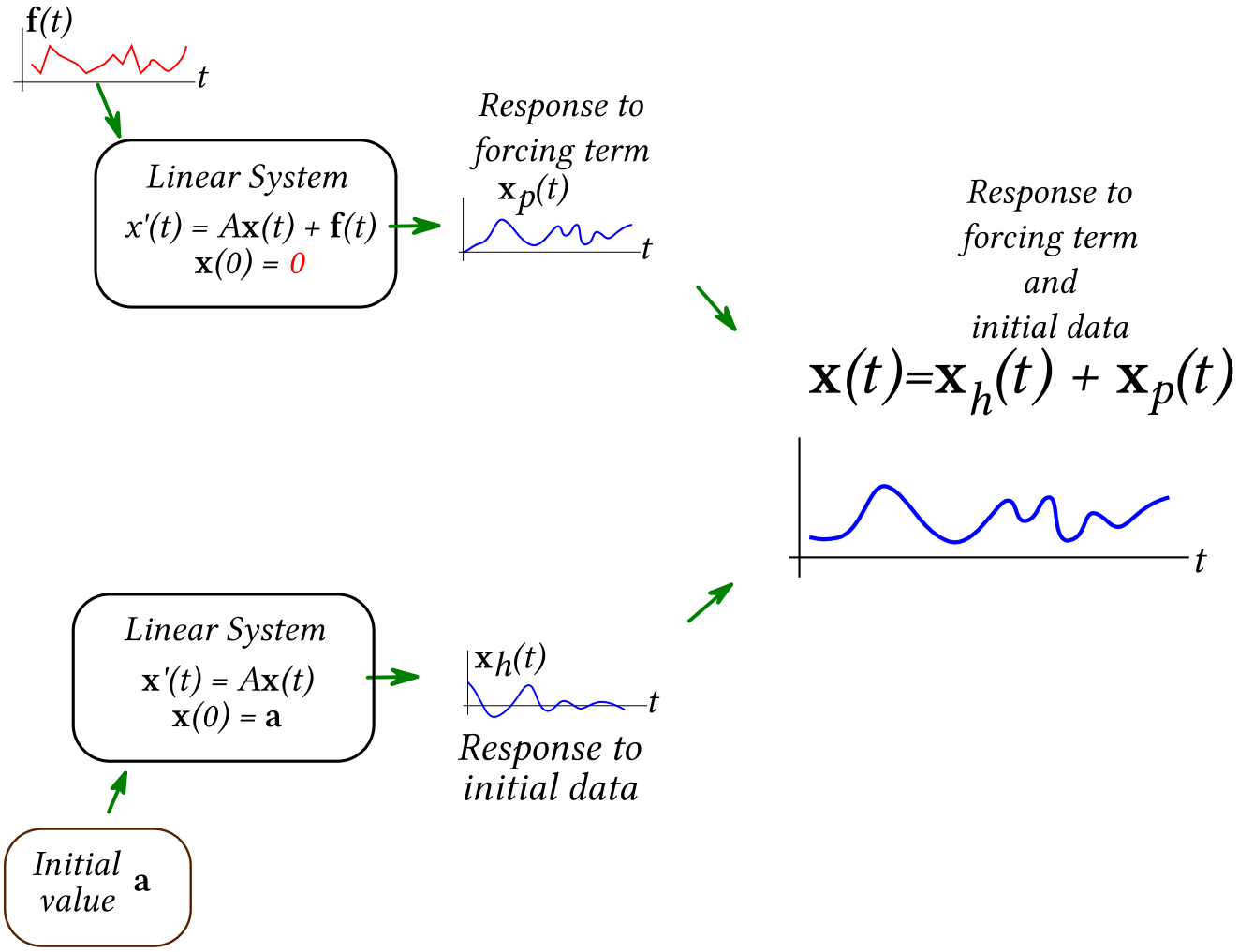
but now we also specify the initial values of both $\mathbf{x}_h(t)$ and $\mathbf{x}_p(t)$: let $\mathbf{x}_h$ be the solution of

$$\mathbf{x}_h'(t) - A\mathbf{x}_h(t) = \mathbf{0}, \qquad \mathbf{x}_h(0) = \mathbf{a} \tag{15}$$

and let $\mathbf{x}_p(t)$ be the solution of

$$\mathbf{x}_p'(t) - A\mathbf{x}_p(t) = \mathbf{f}(t), \qquad \mathbf{x}_p(0) = \mathbf{0} \tag{16}$$

If you think of the solution $\mathbf{x}(t)$ as the "response" to the "input" $\mathbf{f}(t)$, then equation (14) splits the response $\mathbf{x}(t)$ into a component $\mathbf{x}_h(t)$ which is due to the initial conditions, and another component $\mathbf{x}_p(t)$ which is due to the forcing term.


f(t)
t
Linear System
x'(t) = Ax(t) + f(t)
x(0) = 0
Response to forcing term
x_p(t)
t
Response to forcing term and initial data
x(t)=x_h(t) + x_p(t)
t
Linear System
x'(t) = Ax(t)
x(0) = a
x_h(t)
t
Response to initial data
Initial value a

You can find $\mathbf{x}_h(t)$ by solving the homogeneous equation and choosing the constants c_i so that $\mathbf{x}_h(0) = \mathbf{a}$.

You can find the forcing term by using Variation of Constants as above, and requiring $c_1(0) = 0, \ldots, c_n(0) = 0$. Then the equations (12) (and the Fundamental Theorem of Calculus)[1] imply

$$\begin{cases} c_1(t) = \displaystyle\int_0^t c_1'(s)\,ds = \int_0^t e^{-\lambda_1 s} f_1(s)\,ds \\ \qquad\vdots \\ c_n(t) = \displaystyle\int_0^t e^{-\lambda_n s} f_n(s)\,ds \end{cases} \tag{17}$$

Finally you substitute these integrals in the definition (21) of $\mathbf{x}_p(t)$, and you find

$$\mathbf{x}_p(t) = \underbrace{\Big[\int_0^t e^{-\lambda_1 s} f_1(s)\,ds\Big]}_{c_1(t)} e^{\lambda_1 t}\mathbf{v}_1 + \cdots + \underbrace{\Big[\int_0^t e^{-\lambda_n s} f_n(s)\,ds\Big]}_{c_n(t)} e^{\lambda_n t}\mathbf{v}_n \tag{18}$$

[1] The Fundamental Theorem of Calculus says $c(t) - c(0) = \int_0^t c'(s)ds$, so if $c(0) = 0$ then $c(t) = \int_0^t c'(s)ds$

One often rewrites the coefficients of the vectors $\mathbf{v}_i$ as follows:

$$\Big[\underbrace{\int_0^t e^{-\lambda_1 s} f_1(s)\, ds}_{c_1(t)}\Big] e^{\lambda_1 t} = \int_0^t e^{\lambda_1 t} e^{-\lambda_1 s} f_1(s)\, ds = \int_0^t e^{\lambda_1 (t-s)} f_1(s)\, ds. \tag{19}$$

The particular solution with zero initial data is then

$$\mathbf{x}_p(t) = \Big[\int_0^t e^{\lambda_1 (t-s)} f_1(s)\, ds\Big]\mathbf{v}_1 + \cdots + \Big[\int_0^t e^{\lambda_n (t-s)} f_n(s)\, ds\Big]\mathbf{v}_n.$$

3.4 Convolution integrals

Integrals of the form

$$\int_0^t g(t-s)f(s)\,ds$$

are called ***convolution integrals***, and they show up very often when you solve constant coefficient linear differential equations.

The coefficient

$$\int_0^t e^{\lambda_1(t-s)}f_1(s)\,ds.$$

of $\mathbf{v}_1$ in the particular solution $\mathbf{x}_p$ is a convolution integral with $g(t) = e^{\lambda_1 t}$. It has the following interpretation: since $f_1(s)$ is the $\mathbf{v}_1$ component of the input at time s,

> *the* $\mathbf{v}_1$ *component of the solution is an average of the* $\mathbf{v}_1$ *components* $f_1(s)$ *of the input at earlier times* s*.*

The weight with which $f_1(s)$ contributes to the solution at time t only depends on the time delay $t - s$.

4 An example

Let $y(t)$ be the solution of

$$\begin{cases} y''(t) + 3y'(t) + 2y(t) = F(t), \\ y(0) = y'(0) = 0. \end{cases} \tag{20}$$

First we rewrite this as a system of first order equations: let $x_1 = y$, $x_2 = y'$. Then

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - 3x_2 + F(t), \end{cases}$$

so $\mathbf{x} = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ satisfies

$$\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t) \text{ with } A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \text{ and } \mathbf{f}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.$$

The eigenvalues and vectors of A are

$$\lambda_1 = -1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda_2 = -2, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Therefore we should find a particular solution of the form

$$\mathbf{x}_p(t) = c_1(t)e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2(t)e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{21}$$

The initial conditions are

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}.$$

We get these initial conditions if we require

$$c_1(0) = c_2(0) = 0 \tag{22}$$

Substituting $\mathbf{x}_p(t)$ from (21) in $\mathbf{x}' - A\mathbf{x}$ leads to

$$\mathbf{x}_p'(t) - A\mathbf{x}_p(t) = c_1'(t)e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2'(t)e^{-2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{23}$$

by the handy identity (10). To compare this with $\mathbf{f}(t)$ we must write the forcing term $\mathbf{f}(t)$ as a combination of the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$:

$$\mathbf{f}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix} = f_1(t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + f_2(t)\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 \\ -f_1 - 2f_2 \end{pmatrix}.$$

Solve this by row reduction:

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -2 & F(t) \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & F(t) \\ 0 & 1 & -F(t) \end{array}\right] \implies f_1(t) = F(t), f_2(t) = -F(t).$$

Our decomposition of the forcing term $\mathbf{f}(t)$ into the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ is therefore

$$\mathbf{f}(t) = F(t)\mathbf{v}_1 - F(t)\mathbf{v}_2 \tag{24}$$

Now combining (24) and (23) we see that $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$ implies

$$c_1'(t)e^{-t}\mathbf{v}_1 + c_2'(t)e^{-2t}\mathbf{v}_2 = F(t)\mathbf{v}_1 - F(t)\mathbf{v}_2$$

so that

$$c_1'(t)e^{-t} = F(t) \text{ and } c_2'(t)e^{-2t} = -F(t).$$

Solving for c_1' and c_2', and integrating, taking into account that $c_1(0) = c_2(0) = 0$, you get

$$c_1(t) = \int_0^t e^s F(s)\,ds \text{ and } c_2(t) = -\int_0^t e^{2s} F(s)\,ds.$$

Put this back into the solution $\mathbf{x}_p(t)$:

$$\mathbf{x}_p(t) = c_1(t)e^{-t}\mathbf{v}_1 + c_2(t)e^{-2t}\mathbf{v}_2 = \Big[\int_0^t e^{s-t}F(s)\,ds\Big]\mathbf{v}_1 - \Big[\int_0^t e^{2(s-t)}F(s)\,ds\Big]\mathbf{v}_2$$

To see the solution in all its components remember that $\mathbf{v}_1 = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$ and $\mathbf{v}_2 = \left(\begin{smallmatrix}1\\-2\end{smallmatrix}\right)$, so that

$$\mathbf{x}_p(t) = \Big[\int_0^t e^{s-t}F(s)\,ds\Big]\begin{pmatrix}1\\-1\end{pmatrix} - \Big[\int_0^t e^{2(s-t)}F(s)\,ds\Big]\begin{pmatrix}1\\-2\end{pmatrix}.$$

Adding these vectors produces a large formula, whose calculation you normally would try to avoid (see the footnote[2]).

Fortunately we don't need the whole vector $\mathbf{x}(t)$: remember that this problem is about finding the solution $y(t)$ to the second order equation (20), and because of $\mathbf{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$ we only need to find the top component of $\mathbf{x}_p(t)$. It is given by

$$y(t) = \int_0^t e^{s-t} F(s)\, ds - \int_0^t e^{2(s-t)} F(s)\, ds = \int_0^t \{e^{-(t-s)} - e^{-2(t-s)}\} F(s)\, ds.$$

[2]Namely:

$$\mathbf{x}_p(t) = \Big[\int_0^t e^{s-t} F(s)\, ds\Big] \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \Big[\int_0^t e^{2(s-t)} F(s)\, ds\Big] \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \int_0^t e^{s-t} F(s)\, ds \\ -\int_0^t e^{s-t} F(s)\, ds \end{pmatrix} - \begin{pmatrix} \int_0^t e^{2(s-t)} F(s)\, ds \\ -2\int_0^t e^{2(s-t)} F(s)\, ds \end{pmatrix}$$

$$= \begin{pmatrix} \int_0^t e^{s-t} F(s)\, ds - \int_0^t e^{2(s-t)} F(s)\, ds \\ -\int_0^t e^{s-t} F(s)\, ds + 2\int_0^t e^{2(s-t)} F(s)\, ds \end{pmatrix} = \begin{pmatrix} \int_0^t \{e^{s-t} - e^{2(t-s)}\} F(s)\, ds \\ \int_0^t \{-e^{s-t} + 2e^{2(t-s)}\} F(s)\, ds \end{pmatrix}$$

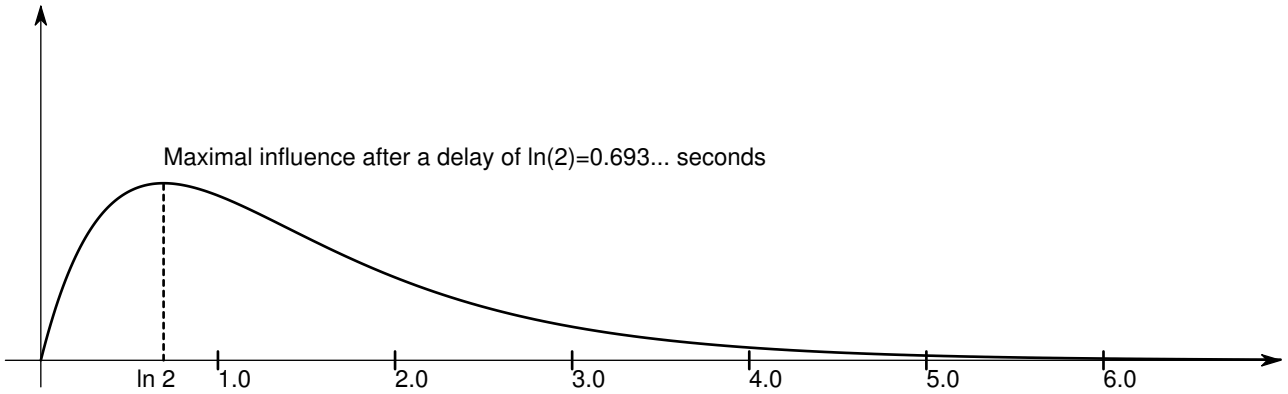
This integral is of the form

$$y(t) = \int_0^t K(t-s)F(s)\,ds$$

where

$$K(\tau) = e^{-\tau} - e^{-2\tau}.$$

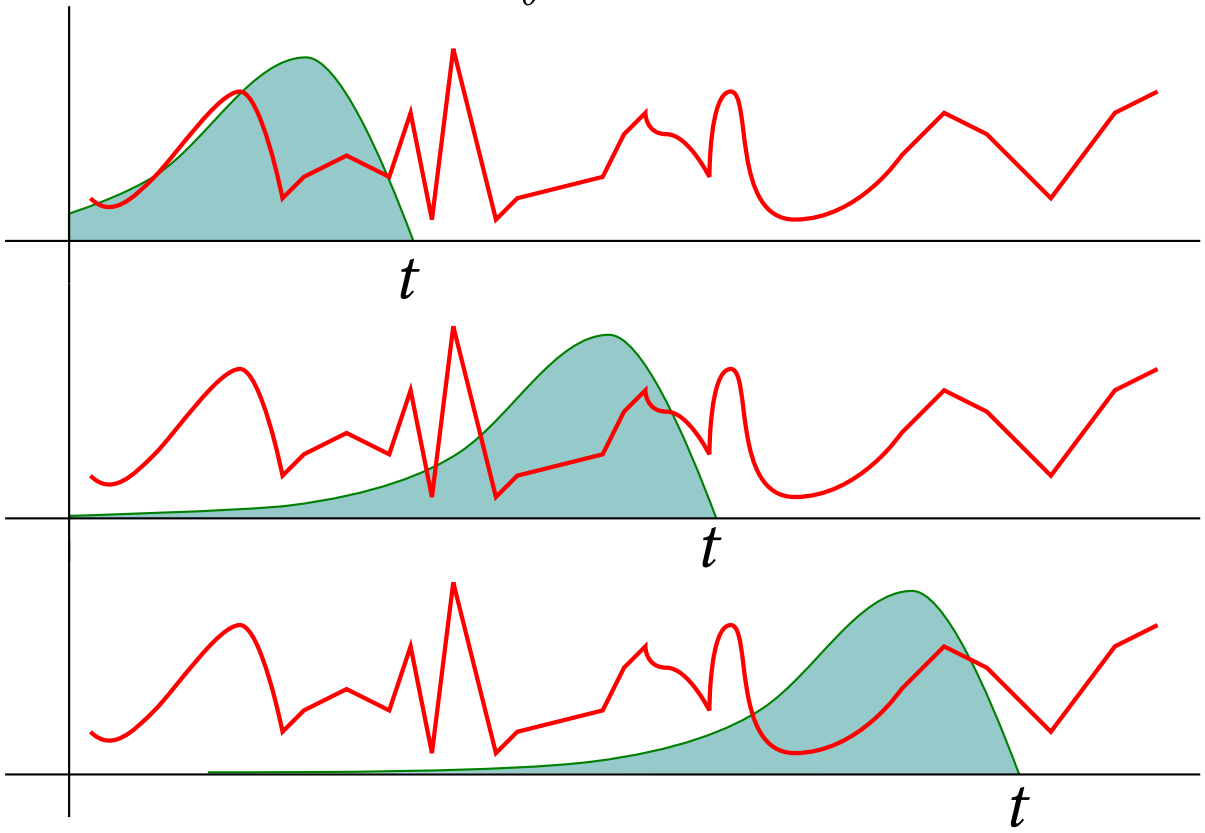
The interpretation of the convolution integral is that the solution $y(t)$ is a response to all the inputs $F(s)$ for $0 < s < t$, and that the contribution of the input $F(s)$ at time s is weighed by a factor $K(t-s)$.

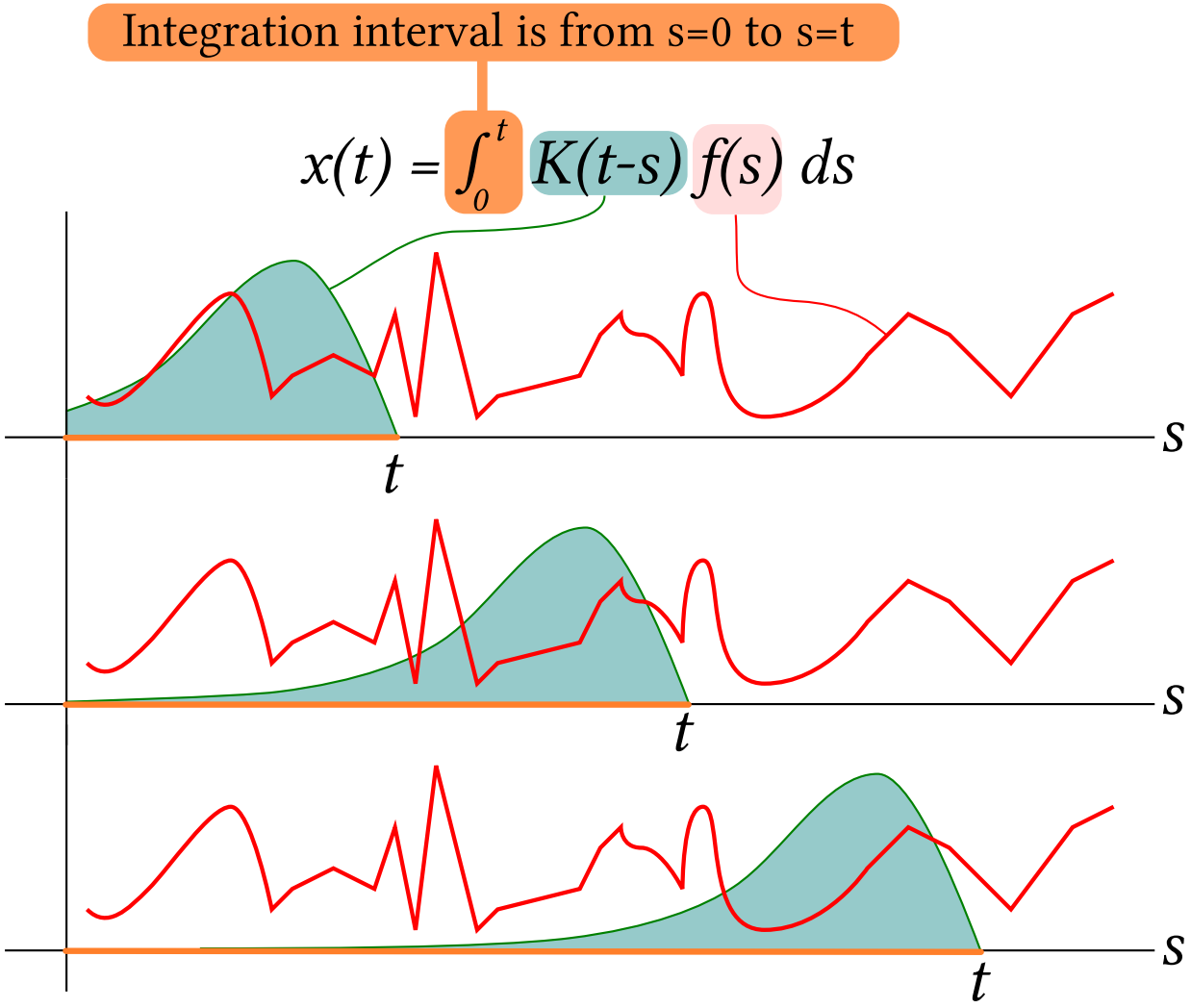


A graph of $K(\tau)$; $K(\tau)$ is maximal at $\tau = \ln 2$

Convolution

$$x(t) = \int_0^t K(t\text{-}s)\, f(s)\, ds$$




Integration interval is from s=0 to s=t
$x(t) = \int_0^t K(t\text{-}s)\, f(s)\, ds$
t
s
t
s
t
s

5 Problems

1. Use the method of variation of Constants to find the solution to $y''+5y'+6y = F(t)$, with $y(0) = y'(0) = 0$. More specifically:
(***i***) rewrite the equation as a system of first order differential equations of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$; what are A and $\mathbf{f}(t)$ in this problem?
(***ii***) Find the eigenvalues and eigenvectors of A.
(***iii***) Find the general solution to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$.
(***iv***) Which kind of solution do you try when you use V.o.C. to find a particular solution?
(***v***) which initial conditions should you impose on $\mathbf{x}$ if you want $y(0) = y'(0) = 0$?
(***vi***) Find the solution $\mathbf{x}(t)$.
(***vii***) Find the solution $y(t)$.

2. Do the same for $y'' - y = F(t)$, with $y(0) = y'(0) = 0$.

3. Use V.o.C. to find the solution to

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{f}(t), \text{ with } A = \begin{pmatrix} -10 & 0 & 0 \\ 10 & -5 & 0 \\ 0 & 5 & -4 \end{pmatrix}, \text{ and } \mathbf{f}(t) = \begin{pmatrix} F(t) \\ 0 \\ 0 \end{pmatrix}.$$

Write a formula for $x_3(t)$ (the third component of $\mathbf{x}(t)$): the answer will be a convo-

lution integral of the form

$$\int_0^t K(t-s)F(s)\,ds$$

Identify $K(t-s)$.

(There is an interpretation of this problem as in example 2 on page 496: in the present problem the inflow of brine into the first tank is given by $F(t)$, rather than just "20" as in the book.)

6 Proof of formula (8)

We have to substitute

$$\mathbf{x}(t) = c_1(t)\mathbf{x}_1(t) + \cdots + c_n(t)\mathbf{x}_n(t)$$

in $\mathbf{x}' - A\mathbf{x}$. The Product Rule tells us that

$$\begin{aligned}\mathbf{x}'(t) = c_1(t)\mathbf{x}_1'(t) + \cdots + c_n(t)\mathbf{x}_n'(t) \\ + c_1'(t)\mathbf{x}_1(t) + \cdots + c_n'(t)\mathbf{x}_n(t).\end{aligned}$$

Since the $\mathbf{x}_j(t)$ are solutions to the homogeneous equation, we get

$$\begin{aligned}\mathbf{x}'(t) = c_1(t)A\mathbf{x}_1(t) + \cdots + c_n(t)A\mathbf{x}_n(t) \\ + c_1'(t)\mathbf{x}_1(t) + \cdots + c_n'(t)\mathbf{x}_n(t).\end{aligned}$$

On the other hand we also have

$$A\mathbf{x}(t) = c_1(t)A\mathbf{x}_1(t) + \cdots + c_n(t)A\mathbf{x}_n(t),$$

and therefore

$$\mathbf{x}'(t) - A\mathbf{x}(t) = c_1'(t)\mathbf{x}_1(t) + \cdots + c_n'(t)\mathbf{x}_n(t). \tag{25}$$

(1i) Choose $x_1 = y$ and $x_2 = y'$ as variables. Then $x_1' = x_2$ and $x_2' = y'' = -6y - 5y' - F(t) = -6x_1 - 5x_2 - F(t)$. The system you have to solve has

$$A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \text{ and } \mathbf{f}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.$$

(1ii) $\det(A - \lambda I) = \lambda^2 + 5\lambda + 6$, $\lambda_1 = -2$, $\mathbf{v}_1 = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$, $\lambda_2 = -3$, $\mathbf{v}_2 = \left(\begin{smallmatrix} 1 \\ -3 \end{smallmatrix}\right)$.

(1iii) $\mathbf{x}_h(t) = c_1 e^{-2t}\mathbf{v}_1 + c_2 e^{-3t}\mathbf{v}_2$.

(1iv) Try $\mathbf{x}(t) = c_1(t)e^{-2t}\mathbf{v}_1 + c_2(t)e^{-3t}\mathbf{v}_2$, i.e. replace the constants c_1, c_2 in the solution to the homogeneous equation by functions.

(1v) $\left(\begin{smallmatrix} x_1(0) \\ x_2(0) \end{smallmatrix}\right) = \left(\begin{smallmatrix} y(0) \\ y'(0) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = \mathbf{0}$

(1vi) Substitute $\mathbf{x}(t) = c_1(t)e^{-2t}\mathbf{v}_1 + c_2(t)e^{-3t}\mathbf{v}_2$ in the LHS of the equation $\mathbf{x}' - A\mathbf{x} = \mathbf{f}(t)$ and use the handy formula:

$$\mathbf{x}' - A\mathbf{x} = c_1'(t)e^{-2t}\mathbf{v}_1 + c_2'(t)e^{-3t}\mathbf{v}_2.$$

We want this to equal $\mathbf{f}(t) = \binom{0}{F(t)}$, and thus we get these equations

$$c_1'(t)e^{-2t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2'(t)e^{-3t}\begin{pmatrix}1\\-3\end{pmatrix} = \begin{pmatrix}0\\F(t)\end{pmatrix},$$

or

$$\begin{pmatrix}1 & 1\\-2 & -3\end{pmatrix}\begin{pmatrix}c_1'(t)e^{-2t}\\c_2'(t)e^{-3t}\end{pmatrix} = \begin{pmatrix}0\\F(t)\end{pmatrix}.$$

Solve for $c_1'(t)e^{-2t}$ and $c_2'(t)e^{-3t}$ by row reducing

$$\left[\begin{array}{cc|c}1 & 1 & 0\\-2 & -3 & F(t)\end{array}\right] \to \left[\begin{array}{cc|c}1 & 1 & 0\\0 & -1 & F(t)\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & F(t)\\0 & 1 & -F(t)\end{array}\right] \implies \begin{array}{l}c_1'(t)e^{-2t} = F(t)\\c_2'(t)e^{-3t} = -F(t)\end{array}$$

so we get

$$c_1'(t) = e^{2t}F(t) \text{ and } c_2'(t) = -e^{3t}F(t).$$

You get c_1 and c_2 by integrating (using the fundamental theorem of calculus)

$$c_1(t) - c_1(0) = \int_0^t e^{2s}F(s)ds, \quad c_2(t) - c_2(0) = -\int_0^t e^{3s}F(s)ds.$$

We need the values of $c_1(0)$ and $c_2(0)$. These come from the initial conditions we have imposed on our particular solution: $\mathbf{x}(0) = \mathbf{0}$ implies $c_1(0) = 0$ and $c_2(0) = 0$. So

$$c_1(t) = \int_0^t e^{2s}F(s)ds, \quad c_2(t) = -\int_0^t e^{3s}F(s)ds.$$

Substitute this in our formula for $\mathbf{x}(t)$:

$$\begin{aligned}\mathbf{x}(t) &= e^{-2t}\int_0^t e^{2s}F(s)ds\ \mathbf{v}_1 - e^{-3t}\int_0^t e^{3s}F(s)ds\ \mathbf{v}_2 \\ &= \int_0^t e^{-2(t-s)}F(s)ds\ \mathbf{v}_1 - \int_0^t e^{-3(t-s)}F(s)ds\ \mathbf{v}_2\end{aligned}$$

This is the answer. You can write it differently by substituting $\mathbf{v}_1 = \left(\begin{smallmatrix}1\\-2\end{smallmatrix}\right)$, $\mathbf{v}_2 = \left(\begin{smallmatrix}1\\-3\end{smallmatrix}\right)$, which leads to

$$\mathbf{x}(t) = \int_0^t e^{-2(t-s)}F(s)ds\ \begin{pmatrix}1\\-2\end{pmatrix} - \int_0^t e^{-3(t-s)}F(s)ds\ \begin{pmatrix}1\\-3\end{pmatrix}$$

(1vii) We have found $\mathbf{x}(t)$. Since $\mathbf{x}(t)$ was defined to be $\left(\begin{smallmatrix}y(t)\\y'(t)\end{smallmatrix}\right)$, we can find $y(t)$ by just taking the first component of $\mathbf{x}(t)$. This leads to

$$y(t) = \int_0^t e^{-2(t-s)}F(s)ds - \int_0^t e^{-3(t-s)}F(s)ds = \int_0^t \{e^{-2(t-s)} - e^{-3(t-s)}\}F(s)ds.$$

(2) Let $x_1 = y$, $x_2 = y'$. Then $\mathbf{x}$ satisfies $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$ with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.$$

The eigenvalues/vectors of A are

$$\lambda_1 = 1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ and } \lambda_2 = -1, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution of the homogeneous equation is

$$\mathbf{x}_h(t) = c_1 e^t \mathbf{v}_1 + c_2 e^{-t} \mathbf{v}_2 = \begin{pmatrix} c_1 e^t + c_2 e^{-t} \\ c_1 e^t - c_2 e^{-t} \end{pmatrix}.$$

The solution of the inhomogeneous equation is

$$\mathbf{x}(t) = c_1(t) e^t \mathbf{v}_1 + c_2(t) e^{-t} \mathbf{v}_2,$$

where we till have to find c_1, c_2. Substituting $\mathbf{x}$ in the equation we get

$$\mathbf{x}'(t) - A\mathbf{x}(t) = c_1'(t) e^t \mathbf{v}_1 + c_2'(t) e^{-t} \mathbf{v}_2 \stackrel{?}{=} \mathbf{f}(t).$$

So we find $c_1'(t)$ and $c_2'(t)$ by solving

$$c_1'(t) e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2'(t) e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ F(t) \end{pmatrix},$$

i.e.

$$\begin{aligned} c_1'(t)e^t + c_2'(t)e^{-t} &= 0 \\ c_1'(t)e^t - c_2'(t)e^{-t} &= F(t). \end{aligned}$$

Solve by row reducing

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & F(t) \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2}F(t) \\ 0 & 1 & -\frac{1}{2}F(t) \end{array}\right]$$

whence

$$c_1'(t) = \tfrac{1}{2}e^{-t}F(t), \text{ and } c_2'(t) = -\tfrac{1}{2}e^{t}F(t).$$

Use the initial conditions $c_1(0) = 0$ and $c_2(0) = 0$ and integrate to get

$$c_1(t) = \tfrac{1}{2}\int_0^t e^{-s}F(s)ds, \qquad c_2(t) = -\tfrac{1}{2}\int_0^t e^{s}F(s)ds.$$

Substitute in our definition of $\mathbf{x}(t)$:

$$\mathbf{x}(t) = \tfrac{1}{2}e^t \int_0^t e^{-s}F(s)ds \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \tfrac{1}{2}e^{-t} \int_0^t e^{s}F(s)ds \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The solution to the original second order equation is $y(t) = x_1(t)$, so it is the first component of the $\mathbf{x}(t)$ we just found. You can write it as one integral involving F, and the integral is of convolution type:

$$\begin{aligned} y(t) &= x_1(t) \\ &= \tfrac{1}{2}e^t \int_0^t e^{-s}F(s)ds - \tfrac{1}{2}e^{-t}\int_0^t e^s F(s)ds \\ &= \tfrac{1}{2}\int_0^t \left\{e^{t-s} - e^{-(t-s)}\right\}F(s)\,ds \\ &= \int_0^t K(t-s)F(s)\,ds \end{aligned}$$

where

$$K(t-s) = \frac{e^{t-s} - e^{-(t-s)}}{2}.$$

Congratulations to anyone who recognizes this as $K(t-s) = \sinh(t-s)$.

(3) First find the eigenvalues/vectors of A:

$$\det(A - \lambda I) = -(\lambda + 10)(\lambda + 5)(\lambda + 4),$$

so the eigenvalues are $\lambda_1 = -10$, $\lambda_2 = -5$, $\lambda_3 = -4$. The eigenvectors turn out to be

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(or multiples of these of course: I chose multiples which don't have fractions).

The general solution to the homogeneous equation is

$$\mathbf{x}_h(t) = c_1 e^{-10t}\mathbf{v}_1 + c_2 e^{-5t}\mathbf{v}_2 + c_3 e^{-4t}\mathbf{v}_3,$$

and to solve the inhomogeneous equation by VoC, we try

$$\mathbf{x}_h(t) = c_1(t) e^{-10t}\mathbf{v}_1 + c_2(t) e^{-5t}\mathbf{v}_2 + c_3(t) e^{-4t}\mathbf{v}_3.$$

We get

$$\mathbf{x}' - A\mathbf{x} = c_1'(t) e^{-10t}\mathbf{v}_1 + c_2'(t) e^{-5t}\mathbf{v}_2 + c_3'(t) e^{-4t}\mathbf{v}_3 \stackrel{?}{=} \mathbf{f}(t).$$

To find $c_1'(t)e^{-10t}$, $c_2'(t)e^{-5t}$, and $c_3'(t)e^{-4t}$, we row reduce

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & F(t) \\ -6 & 1 & 0 & 0 \\ 5 & -5 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & 0 & F(t) \\ 0 & 1 & 0 & 2F(t) \\ 0 & -5 & 1 & -\frac{5}{3}F(t) \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3}F(t) \\ 0 & 1 & 0 & 2F(t) \\ 0 & 0 & 1 & \frac{25}{3}F(t) \end{array}\right]$$

So we find

$$c_1'(t)e^{-10t} = \frac{1}{3}F(t), \quad c_2'(t)e^{-5t} = 2F(t), \quad c_3'(t)e^{-4t} = \frac{25}{3}F(t).$$

Integrate, using the initial conditions $\mathbf{x}(0) = \mathbf{0} \implies c_1(0) = c_2(0) = c_3(0) = 0$, and substitute in the definition of $\mathbf{x}(t)$ to get

$$\mathbf{x}(t) = \frac{1}{3}\int_0^t e^{-10(t-s)}F(s)\,ds\mathbf{v}_1 + 2\int_0^t e^{-5(t-s)}F(s)\,ds\mathbf{v}_2 + \frac{25}{3}\int_0^t e^{-4(t-s)}F(s)\,ds\mathbf{v}_3.$$

The third components of the eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are 5, -5, and 1, so the third component of $\mathbf{x}$ is

$$\begin{aligned} x_3(t) &= \frac{1}{3}\int_0^t e^{-10(t-s)}F(s)\,ds \times 5 + 2\int_0^t e^{-5(t-s)}F(s)\,ds \times (-5) + \frac{25}{3}\int_0^t e^{-4(t-s)}F(s)\,ds \times 1 \\ &= \int_0^t \{\frac{5}{3}e^{-10(t-s)} - 10e^{-5(t-s)} + \frac{25}{3}e^{-4(t-s)}\}F(s)\,ds. \end{aligned}$$