MATH 319 Final Exam—Spring 1998

A. Power series

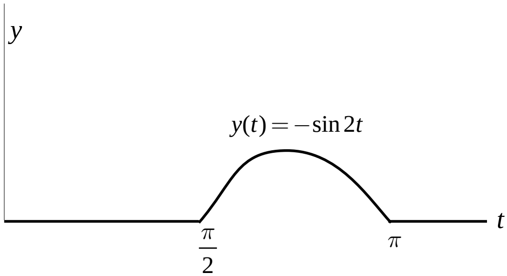
Is $x=0$ a regular or regular singular point for the following equation

$$\frac{y''}{1+x^2} - \frac{12}{x^2}y = 0 \ ?$$

Find *one nonzero solution* of the equation in part A by the appropriate power series method (find the recurrence relation, and a formula for the coefficient a_n).

B. Laplace transform

1. Compute the Laplace transform of the function $y(t)$ with the depicted graph:



Let $f(t)$ be a function whose Laplace transform is given by

$$\mathrm{L}\{f(t)\} = F(s) = \frac{e^{-\pi s/2}}{s+1} + \frac{e^{-\pi s}}{s^2+1}$$

Compute $f(\frac{3}{4}\pi)$.

2. Use convolutions to find the solution to the following equation (you may leave an integral in your answer!)

$$\begin{cases} y''(t) - 3y'(t) + 2y(t) = f(t) \\ y(0) = y'(0) = 0 \end{cases}$$

3. Find the Laplace transform of $y(t)$ where $x(t)$ and $y(t)$ are the solution of the following system:

$$\begin{cases} x'(t) = 2x(t) - 2y(t) + f(t) \\ y'(t) = 3x(t) - 5y(t) \end{cases} \qquad x(0)=0; \quad y(0)=15$$

(The Laplace transform $F(s)$ of the unknown function $f(t)$ may appear in you answer.)

C. Eigenvector & value analysis

1. Let A be a 2×2 matrix whose eigenvalues&vectors are $\lambda = 2, \ \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mu = -3, \ \mathbf{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Sketch the phase portrait of the linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$; in particular draw the trajectory that goes through a point on the y-axis (draw but do not compute!).

2. Consider the matrix $B = \begin{pmatrix} -1 & -1 \\ 1 & -4 \end{pmatrix}$. If you compute them you find that the eigenvalues of B are $\lambda_{\pm} = -2 \pm i$.

Again, sketch the phase portrait of the linear system $\mathbf{x}'(t) = B\mathbf{x}(t)$ and in particular draw (but don't compute) the trajectory that goes through a point on the y-axis.

D. Phase plane and linearized stability

Consider the system of differential equations

$$(\S) \qquad \begin{cases} x'(t) = 6x - 2x^2 - xy \\ y'(t) = 9y - xy - 2y^2 \end{cases}$$

1. Find all equilibrium points of the system (§). Also draw the "direction field," i.e. the regions where $x' > 0, < 0$ and also the regions where $y' > 0, < 0$.

2. The system (§) has an equilibrium (x_0, y_0) in the first quadrant (i.e. with both $x_0 > 0$ and $y_0 > 0$). Use the method of linearized stability to determine whether this equilibrium is stable or not.