MATH 319, FINAL EXAM, FRIDAY 12/20/96.

Name:

Score	4:
1:	5:
2:	Total:
3:	

1. Consider the following differential equation

$$(2y + \alpha \tan x)\frac{dy}{dx} + 1 + \frac{y}{\cos^2 x} = 0 \tag{1}$$

in which α is a constant.
(a) [9pts] For which value of α is (1) an exact equation?
(b) [9pts] Solve the differential equation you get by substituting the value for α which you found in part (a) into equation (1).

2. [10pts] The differential equation

$$(x^2 + x^4)y''(x) - (2x + 4x^3)y'(x) + (2 + 6x^2)y(x) = 0 \tag{2}$$

has $y_1(x) = x^2$ as a solution. Find the general solution of (2).

3. Consider the differential equation

$$\frac{d^2y}{dx^2} + \left(\frac{2}{x} - x\right)\frac{dy}{dx} + \lambda y = 0 \tag{3}$$

in which λ is some constant.
(a) [4pts] Determine whether $x = 0$ is an ordinary, regular singular or irregular singular point.
(b) [12pts] Find at least one powerseries type solution of (3).
(c) [2pts] For which value of λ is the solution you found in (b) a polynomial?

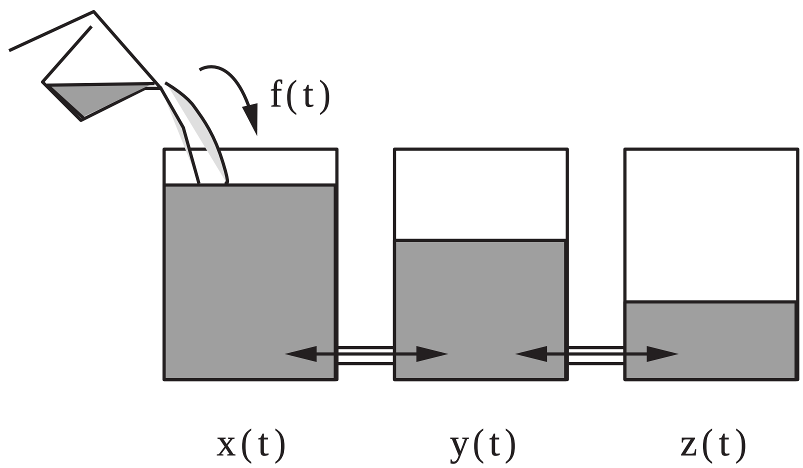
4. **(a)** [9pts] Find the Laplace transform of the following function

$$f(t) = \begin{cases} 25t & \text{for } 0 \le t \le 1 \\ \frac{25}{e}e^{-t} & \text{for } t \ge 1. \end{cases}$$

(b) [7pts] The Laplace transform of a function $f(t)$ is given by

$$F(s) = \frac{G(s)}{s-3}$$

where $G(s)$ is the Laplace transform of some unknown function $g(t)$. Find a formula for $f(t)$.
(c) [2pts] Prove: If the unknown function $g(t)$ in part (b) is nonnegative for all t, then so is the function $f(t)$.



5. Molasses is being poured into the vessel on the left at a rate of $f(t)$ cc per second; the vessels are connected by tubes, as shown, and the rate at which molasses flows through such a tube is proportional to the difference in molasses levels on either side of the tube. All this leads to the following set of differential equations for the levels of molasses $x(t)$, $y(t)$ and $z(t)$.

$$x'(t) = -K_1(x(t) - y(t)) + f(t) \tag{4a}$$
$$y'(t) = K_1(x(t) - y(t)) - K_2(y(t) - z(t)) \tag{4b}$$
$$z'(t) = K_2(y(t) - z(t)) \tag{4c}$$

On this exam we will assume that the proportionality constants K_1 and K_2 are given by

$$K_1 = 1, \qquad K_2 = 1.$$

(a) [10pts] Suppose that initially vessel x is full ($x(0) = 1$) and the other two vessels are empty ($y(0) = z(0) = 0$). Assume that *no* molasses is being poured into the system (i.e. $f(t) = 0$). Then solve these equations for $z(t)$.
(b) [2pts] How much is $\lim_{t\to\infty} z(t)$ for the $z(t)$ you found in part (a)?
(c) [8pts] Assume now that initially all vessels are empty, i.e. $x(0) = y(0) = z(0) = 0$, but that molasses gets poured into vessel x at rate $f(t)$. Again solve the differential equations for $z(t)$ (you may leave an integral and of course the unknown function $f(t)$ in your answer.)