319 Notes on Complex Exponentials and solving Second Order Diffeqs.

1. About Complex Conjugates.

For a quick introduction or refresher on complex numbers read appendix H of the calcuilus text book "*Calculus, Early Transcendentals*" by J.Stewart, which is used in Madison for the 221-222-234 sequence.

If $z = x + iy$ is a complex number then its complex conjugate is $\bar{z} = x - iy$. A basic formula for the complex conjugate is

Example: $\overline{2-3i} = 2+3i$

$$z + \bar{z} = 2\mathrm{Re}(z), \qquad z - \bar{z} = 2\mathrm{Im}(z) \tag{1}$$

Other basic properties of the complex conjugate which are proved in Stewart's book:

$$\overline{z+w} = \bar{z} + \bar{w}, \qquad \overline{zw} = \bar{z}\,\bar{w}. \tag{2}$$

$(2+3i)(1-i) = 5+i$ so $(2-3i)(1+i) = 5-i$ check it!

The complex exponential also behaves nicely with respect to complex conjugation:

$$\overline{e^z} = e^{\bar{z}} \tag{3}$$

Here's why: write $z = x + iy$, then

$$\begin{aligned}
\overline{e^z} &= \overline{e^{x+iy}} \\
&= \overline{e^x \cos y + ie^x \sin y} && \text{def of } e^{x+iy} \\
&= e^x \cos y - ie^x \sin y && \text{def of conjugate} \\
&= e^x \cos(-y) + ie^x \sin(-y) && \text{property of sin, cos} \\
&= e^{x-iy} \\
&= e^{\bar{z}}.
\end{aligned}$$

2. Using Complex Exponential to solve 2nd order equations.

By way of example let's solve problem 21 in §3.4 of the text book using complex exponentials. The problem is

$$y'' + y' + \frac{5}{4}y = 0 \tag{4}$$

$$y(0) = 3 \tag{5}$$

$$y'(0) = 1 \tag{6}$$

Step 1 is to try $y = e^{rt}$ and get the characteristic equation for the exponent r,

$$r^2 + r + \frac{5}{4} = 0 \tag{7}$$

whose roots are

$$r_1 = -\frac{1}{2} + i, \qquad r_2 = -\frac{1}{2} - i. \tag{8}$$

Thus the general solution is

$$y(t) = c_1 e^{(-\frac{1}{2}+i)t} + c_2 e^{(-\frac{1}{2}-i)t}. \tag{9}$$

Here the coefficients c_1 and c_2 are complex numbers. **To get a solution $y(t)$ which is real we take c_2 to be the complex conjugate of** c_1, i.e. we set

$$c_1 = A + iB \text{ and } c_2 = A - iB \tag{10}$$

where A and B are real numbers. This choice of c_1 and c_2 always leads to a real valued solution with two *real* constants A and B in it. You can see this by rewriting (9) as

$$y(t) = c_1 e^{(-\frac{1}{2}+i)t} + c_2 e^{(-\frac{1}{2}-i)t} \tag{11}$$

(use $c_2 = \overline{c_1}$ and $e^{(-\frac{1}{2}-i)t} = \overline{e^{(-\frac{1}{2}+i)t}}$)

$$= c_1 e^{(-\frac{1}{2}+i)t} + \overline{c_1 e^{(-\frac{1}{2}+i)t}} \tag{12}$$

$$= 2\mathrm{Re}\left\{c_1 e^{(-\frac{1}{2}+i)t}\right\} \tag{13}$$

Step 2 is to find A and B so our solution fits the initial data. We do this using (13). In this equation we put $t = 0$ and find that $y(0) = 3$ leads to

$$3 = y(0) = 2\mathrm{Re}(c_1 e^0) = 2\mathrm{Re}(c_1) = 2A \text{ so that } A = \frac{3}{2}. \tag{14}$$

Next compute $y'(t)$,

$$y'(t) = \frac{d}{dt} 2\mathrm{Re}\left\{c_1 e^{(-\frac{1}{2}+i)t}\right\} = 2\mathrm{Re}\left\{c_1 (-\frac{1}{2}+i) e^{(-\frac{1}{2}+i)t}\right\} \tag{15}$$

so that $y'(0) = 1$ tells us

$$\begin{aligned} 1 = y'(0) &= 2\mathrm{Re}\left\{c_1(-\frac{1}{2}+i)\right\} \\ &= 2\mathrm{Re}\left\{(A+iB)(-\frac{1}{2}+i)\right\} \\ &= 2\mathrm{Re}\left\{-\frac{A}{2} - B + i(\cdots)\right\} \qquad \text{(Don't compute imaginary part)} \\ &= -A - 2B. \end{aligned}$$

Since we already have $A = 3/2$ this tells us that $B = -5/4$. The solution is therefore

$$y(t) = 2\mathrm{Re}\left\{(\frac{3}{2} - \frac{5}{4}i) e^{(-\frac{1}{2}+i)t}\right\}$$

$$\boxed{y(t) = \mathrm{Re}\left\{(3 - \frac{5}{2}i) e^{(-\frac{1}{2}+i)t}\right\}} \tag{16}$$

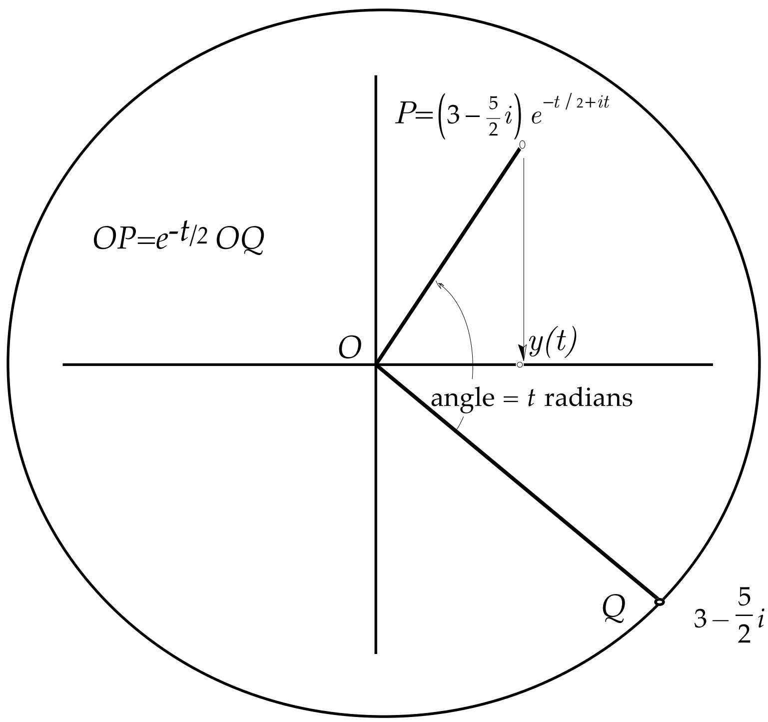
Remember: to multiply complex numbers you multiply their lengths and add their arguments.

Here's a picture of the solution, i.e. a graphical interpretation of (16), which came about as follows:

First draw the complex number $3 - \frac{5}{2}i$ and then multiply this number with $e^{-t/2-it}$, i.e. rotate it counter clockwise by an angle t, and change the length by a factor $e^{-t/2}$. The result is the point or complex number P, and the real part of this complex number is the solution $y(t)$ according to (16)

Some Questions

1. Let $z = 2 + \frac{\pi}{4}i$. Compute e^z, $e^{\bar{z}}$ and $\overline{e^z}$ and draw these points in one picutre.
2. At which time does the solution become zero? In other words what is the first $t > 0$ for which $y(t) = 0$. (*Hint:* Where would the point P in the figure have to be for $y(t)$ to vanish?)



3. Do Problem 18 from §3.4 in the same way as above. Include a drawing of the complex form of the solution, and find the first zero of the solution (i.e. solve $y(t) = 0$).