Wallis' Product

Consider the integral

$$I_n = \int_0^\pi (\sin x)^n \, dx.$$

Integration by parts gives

$$\begin{aligned} I_n &= \int_0^\pi \sin^{n-1} x \ \sin x \, dx \\ &= \left[-\sin^{n-1} x \cos x \right]_0^\pi + \int_0^\pi (n-1) \cos^2 x \sin^{n-2} x \, dx \\ &= (n-1) \int_0^\pi \sin^{n-2} x \, dx + (n-1) \int_0^\pi \sin^n x \, dx \\ &= (n-1) I_{n-2} + (n-1) I_n \end{aligned}$$

where we have used $\cos^2 x = 1 - \sin^2 x$. The last line is an equation for I_n, which implies the following reduction formula

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{0.1}$$

If we know I_0 and I_1 then repeated use of the reduction formula gives us I_n for all $n \geqslant 0$.

Theorem 0.1. *For any integer $k \geqslant 0$ one has*

$$\begin{aligned} I_{2k} &= \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{1}{2} \cdot \pi \\ I_{2k+1} &= \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{2}{3} \cdot 2 \end{aligned}$$

Proof. First you compute $I_0 = \pi$, and $I_1 = 2$. Then you use mathematical induction and 0.1 to prove the general case. □

Wallis observed that for all $x \in [0, \pi]$ one has $0 \leqslant \sin x \leqslant 1$ and thus for any integer $n \geqslant 0$ one has

$$(\sin x)^{n+1} \leqslant (\sin x)^n \ \text{ for all } x \in [0, \pi],$$

and thus one has

$$\int_0^\pi (\sin x)^{n+1} \, dx \leqslant \int_0^\pi (\sin x)^n \, dx, \ \text{ i.e. } \ I_{n+1} \leqslant I_n$$

for all $n = 0, 1, 2, \ldots$In particular, using the values of I_n provided by the theorem, we conclude from

$$I_{2k+1} \leqslant I_{2k} \leqslant I_{2k-1}$$

that

$$\frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{2}{3} \cdot 2 \leqslant \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{1}{2} \cdot \pi \leqslant \frac{2k-2}{2k-1} \cdots \frac{2}{3} \cdot 2$$

Divide by 2 and divide again by $\frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{3}{2}$ to get

$$\alpha_k \stackrel{\text{def}}{=} \frac{(2k)^2}{2k+1} \cdot \frac{(2k-2)^2}{(2k-1)^2} \cdots \frac{4^2}{3^2} \cdot \frac{2^2}{1} \leqslant \frac{\pi}{2} \leqslant 2k \frac{(2k-2)^2}{(2k-1)^2} \cdots \frac{4^2}{3^2} \cdot \frac{2^2}{1} \stackrel{\text{def}}{=} \beta_k$$

The ratio between the left hand side and the right hand side is

$$\frac{\beta_k}{\alpha_k} = \frac{2k+1}{2k} = 1 + \frac{1}{2k}.$$

It follows that

$$\lim_{k \to \infty} 2k \left\{ \frac{(2k-2)(2k-4) \cdots 6 \cdot 4 \cdot 2}{(2k-1)(2k-3) \cdots 3 \cdot 1} \right\}^2 = \frac{\pi}{2},$$

or,

$$\boxed{\frac{\pi}{2} = \lim_{k \to \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdots \frac{2k-2}{2k-3} \cdot \frac{2k-2}{2k-1} \cdot \frac{2k}{2k-1}}$$