Math 276 notes / Spring 2004

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

1. Solving 2×2 systems of linear equations

From algebra you know how to solve a linear system of equations

$$\begin{cases} ax + by = p \\ cx + dy = q \end{cases} \tag{1}$$

in two unknowns x and y. For instance, you could multiply the first equation with d and the second with c, and subtract, with result $(ad - bc)x = pd - qc$. This gives you x. A similar trick will give you y. In the end the solution is given by

$$x = \frac{dp - bq}{ad - bc}, \quad y = \frac{aq - cp}{ad - bc}. \tag{2}$$

There is a special notation for the quantity $ad - bc$ which occurs in the denominator. It is called the *determinant* of the system, and is written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \stackrel{\text{def}}{=} ad - bc. \tag{3}$$

With this notation we can reformulate the above as follows

Theorem 1 (“Cramer’s rule,” the 2×2 case)**.** *If* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ *then the system of equations* (1) *has a solution for any given* $p, q \in \mathbb{R}$. *This solution is given by*

$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

2. The Wronskian and Abel’s theorem

Consider the second order linear differential equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{4}$$

and its associated *homogeneous equation*

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0. \tag{5}$$

Here, and in the following all functions are assumed to be defined on some interval $x_1 < x < x_2$.

Definition. If y_1 and y_2 are solutions of the homogeneous equation, then their *Wronskian* is defined to be the function

$$W(x) \stackrel{\text{def}}{=} W(y_1, y_2; x) \stackrel{\text{def}}{=} y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Using the determinant notation we have therefore defined the Wronskian to be

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Theorem 2 (Niels Henrik Abel, (1802–1829))**.** *The Wronskian of two solutions of the linear homogeneous differential equation* (5) *satisfies*

$$\frac{dW}{dx} = -a(x)W(x).$$

Hence $W(x)$ is given by

$$W(x) = W(x_0)e^{\int_{x_0}^{x} a(x')dx'} \tag{6}$$

3. The Method of Variation of Constants

To solve the inhomogeneous equation (4) one can use the method of *Variation of Constants* (or "variation of parameters"). In this method one assumes that the solution y is given by

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x), \tag{7}$$

and that the functions c_1 and c_2 satisfy

$$y'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x), \tag{8}$$

Such functions c_1 and c_2 always exist, at least if the Wronskian of the two solutions y_1 and y_2 is nonzero (see problem 4). If this is so, then one has

$$\begin{aligned} y''(x) &= c_1'(x)y_1'(x) + c_2'(x)y_2'(x) && +c_1(x)y_1''(x) + c_2(x)y_2''(x) \\ a(x)y'(x) &= && a(x)c_1(x)y_1'(x) + a(x)c_2(x)y_2'(x) \\ b(x)y(x) &= && b(x)c_1(x)y_1(x) + b(x)c_2(x)y_2(x) \end{aligned}$$

Keep in mind that y_1 and y_2 both satisfy the homogeneous equation, and add vertically. You find that

$$f(x) = c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{9}$$

This gives us one equation for $c_1'(x)$ and $c_2'(x)$. To get a second equation we differentiate (7), applying the product rule, and combine the result with (8). One gets

$$0 = c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{10}$$

Equations (9) and (10) together form a system of two equations for the unknowns $c_1'(x)$ and $c_2'(x)$, namely

$$\begin{cases} y_1(x)c_1'(x) + y_2(x)c_2'(x) = 0 \\ y_1'(x)c_1'(x) + y_2'(x)c_2'(x) = f(x) \end{cases} \tag{11}$$

If the Wronskian $W(x) = y_1'(x)y_2(x) - y_1(x)y_2'(x)$ is nonzero, then one can solve this system for $c_1'(x)$ and $c_2'(x)$. Integrating $c_1'(x)$ and $c_2'(x)$ then gives $c_1(x)$ and $c_2(x)$, and from there you get the solution $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$.

When you work this out, you get

$$c_1'(x) = \frac{-y_2(x)f(x)}{W(x)}, \quad c_2'(x) = \frac{y_1(x)f(x)}{W(x)}, \tag{12}$$

where $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of y_1 and y_2. Thus the solution of the inhomogeneous equation (4) is given by

$$y(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\, dx. \tag{13}$$

Both indefinite integrals contain a constant, so the general solution we have found has two undetermined constants in it.

Besides giving us a method for solving the inhomogeneous equation, this computation also lets us prove a uniqueness theorem for the homogeneous equation.

Theorem 3. *Let $y_1, y_2 : (x_1, x_2) \to \mathbb{R}$ be two solutions of the homogeneous equation (5), for which the Wronskian $W(x)$ does not vanish. Then the general solution to the homogeneous equation (5) is*

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x). \tag{14}$$

Proof. The homogeneous equation is just a special case of the inhomogeneous equation where $f(x)$ hapens to vanish. So we can apply the method of Variation of Constants to get the general solution to the homogeneous equation by setting $f = 0$ in (13). The two integrals that appear in (13) now are:

$$\int \frac{y_2(x) f(x)}{W(x)}\, dx = \int 0\, dx = C_1, \quad \int \frac{y_1(x) f(x)}{W(x)}\, dx = \int 0\, dx = C_2.$$

Hence (13) says that the general solution is indeed given by (14). □

4. Linearity and the Superposition Principle

We abbreviate the lefthand side of the differential equation (4) by

$$\mathcal{L}[y] = y''(x) + a(x)y'(x) + b(x)y(x).$$

Thus the equations (4) and (5) can be written concisely as follows:

$$\text{inhomogeneous:} \quad \mathcal{L}[y] = f$$
$$\text{homogeneous:} \quad \mathcal{L}[y] = 0.$$

The expression $\mathcal{L}[y]$ is "linear in y," which, by definition, means that *for any two functions y_1 and y_2, and any two numbers c_1 and c_2 one has*

$$\mathcal{L}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{L}[y_1] + c_2 \mathcal{L}[y_2]. \tag{15}$$

Just as for 1st order equations one has a Superposition Principle.

Theorem 4 (Superposition Principle)**.**

(i) *If y_1 and y_2 are solutions of the homogeneous equation, then so is any linear combination $y = c_1 y_1 + c_2 y_2$ ($c_1, c_2 \in \mathbb{R}$).*

(ii) *If y_1 and y_2 are solutions of the inhomogeneous equations $\mathcal{L}[y_1] = f_1$ and $\mathcal{L}[y_2] = f_2$ respectively, then the linear combination $y = c_1 y_1 + c_2 y_2$ $(c_1, c_2 \in \mathbb{R}$ constants$)$ satisfies $\mathcal{L}[y] = c_1 f_1 + c_2 f_2$.*

(iii) *If y_1 and y_2 are solutions to the same inhomogeneous equation, i.e. if $\mathcal{L}[y_1] = f$ and $\mathcal{L}[y_2] = f$, then their difference $y_h = y_1 - y_2$ satisfies the homogeneous equation: $\mathcal{L}[y_h] = 0$.*

5. Constant coefficient equations

There is no formula that gives you the general solution to the homogeneous equation for an arbitrary second order linear equation. But if the coefficients $a(x)$ and $b(x)$ are constant, such a formula does exist.

Consider the differential equation

$$y'' + py' + qy = 0, \tag{16}$$

where $p, q \in \mathbb{R}$ are constants. To solve this equation one looks for exponential functions which satisfy the equation. So set $y = e^{rx}$ for some constant r, and see if (16) holds:

$$y'' + py' + qy = r^2 e^{rx} + pre^{rx} + qe^{rx} = \left(r^2 + pr + q\right)e^{rx}.$$

Since $e^{rx} \neq 0$ no matter what r and x are (even if they are complex numbers!) we see that $y = e^{rx}$ is a solution of the homogeneous equation *if and only if* r satisfies the quadratic equation

$$r^2 + pr + q = 0. \tag{17}$$

There are now three cases:

$\boxed{p^2 - 4q > 0}$ In this case the characteristic equation has two real roots, r_1 and r_2, and we get two solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ of the homogeneous equation. it follows from the superposition principle that

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a solution of the homogeneous equation for any $c_1, c_2 \in \mathbb{R}$.

$\boxed{p^2 - 4q < 0}$ In this case the characteristic equation has two complex roots, which we write as

$$r_\pm = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\Omega,$$

where $\alpha = -p/2$ and $\Omega = \frac{1}{2}\sqrt{4q - p^2}$ are real numbers, and $\Omega > 0$. The solutions in exponential form are now

$$y_\pm(x) = e^{r_\pm x} = e^{(\alpha \pm i\Omega)x} = e^{\alpha x} e^{\pm i\Omega x} = e^{\alpha x}\left(\cos \Omega x \pm i \sin \Omega x\right)$$

These solutions are compex valued. To get real valued solutions one forms these linear combinations:

$$y_1(x) = \frac{1}{2}\left(y_+(x) + y_-(x)\right) = e^{\alpha x} \cos \Omega x$$

$$y_2(x) = \frac{1}{2i}\left(y_+(x) - y_-(x)\right) = e^{\alpha x} \sin \Omega x$$

Thus we find the following solutions for the homogeneous equation in this case:

$$y(x) = Ay_1(x) + By_2(x) = e^{\alpha x}\left(A \cos \Omega x + B \sin \Omega x\right).$$

$\boxed{p^2 - 4q = 0}$ In this last case the characteristic equation has one double root, namely $r = -p/2$. There is therefore only one exponential function $y(x) = e^{rx}$ which satisfies the equation. It turns out that in this case there is another solution which is not exponential, namely, xe^{rx}. So in this case we have the following solution to our constant coefficent equation (16),

$$y(x) = c_1 e^{rx} + c_2 xe^{rx} = \left(c_1 + c_2 x\right)e^{rx}.$$

6. Questions and Problems

1 Derive the solutions in (2) (i.e. check those formulas.)

2 Solve

$$\begin{cases} x+iy=2+6i \\ ix+y=-4 \end{cases} \quad \text{and} \quad \begin{cases} (2+i)x-2y=2+6i \\ -2x+(2-i)y=-4 \end{cases}$$

"by hand," and again using Cramer's rule.

3 Prove Abel's theorem: verify that the Wronskian really does satisfy $W'(x)=-a(x)W(x)$.

4 Suppose y_1 and y_2 are two solutions of the homogeneous equation (5) whose Wronskian does not vanish. *Show:* If y is an arbitrary differentiable function, then there always exist functions $c_1(x)$ and $c_2(x)$ such that (7) and (8) hold. (Can you write down a formula for c_1 and c_2 in terms of y, y', y_1, y_2, y_1' and y_2'?)

5 Which are the known functions, and which are the unknown functions in (11)?

6 Prove that (10) does indeed follow from the assumptions (7) and (8).

7 (i) State the definition of the statement $\mathcal{L}[y]$ is linear in y."
(ii) Show that the expression $\mathcal{L}[y]=y''(x)+a(x)y'(x)+b(x)y(x)$ is indeed linear in y.

8 Consider the operator $\mathcal{M}$ defined by $\mathcal{M}[y]=\frac{dy}{dx}+y^2$.
(i) Compute $\mathcal{M}[y]$ when y is the function $y(x)=\sin x$. Do the same for $y=2\sin x$.
(ii) Is $\mathcal{M}[y]$ linear in y?

9 Prove Theorem 4!!

10 Find the general solutions to the following diffeqs:

$$2y''(x)+3y'(x)+y(x)=0 \qquad y''(x)-16y(x)=0$$
$$y''(x)+Ay'(x)+y(x)=0 \qquad y''(x)-y'(x)+Ay(x)=0$$

where $A>0$ is some constant.

11 Explain how you can use the Superposition Principle (Theorem 4) to find a particular and from there the general solution to the differential equations

$$y''(x)+\omega^2y(x)=1+x+x^2$$
$$y''(x)+2y'(x)-y(x)=x^2+\sin(Ax)$$
$$6y''(x)+5y'(x)+y(x)=e^{Ax}$$

Here A and ω are positive constants.

12 Use Variation of Constants to find the general solution of the following equations:

$$y''(x)-y(x)=x$$
$$y''(x)+4y(x)=\sin Ax \qquad A>0 \text{ is some constant.}$$
$$y''(x)-y'(x)=e^{i\omega x} \qquad \omega>0 \text{ is a constant.}$$