Math 276 notes / Spring 2004

LINEAR DIFFERENTIAL EQUATIONS

Contents

1. FIRST ORDER EQUATIONS

1.1. Solution Method

Suppose A, B and C are continuous functions on an interval $a < x < b$, and suppose $A(x) \neq 0$ for all $x \in (a, b)$.

To solve a differential equation of the form

$$A(x)\frac{dy}{dx} + B(x)y(x) = C(x), \tag{1}$$

you first divide both sides by $A(x)$, to get

$$\frac{dy}{dx} + P(x)y(x) = Q(x), \tag{2}$$

where $P(x) = B(x)/A(x)$ and $Q(x) = C(x)/A(x)$.

Next, you multiply the equation with a function $m : (a, b) \to \mathbb{R}$ which is to be determined later:

$$m(x)\frac{dy}{dx} + m(x)P(x)y(x) = m(x)Q(x), \tag{3}$$

We require that m satisfies

$$\frac{dm}{dx} = m(x)P(x). \tag{4}$$

Any function $m(x)$ which satisfies this equation is called an *integrating factor*.

Equation (4) is a separable differential equation for m, so you can always solve it. One solution is given by

$$m(x) = e^{\int P(x)dx} \tag{5}$$

Let m be an integrating factor. Then (3) implies

$$m(x)\frac{dy}{dx} + \frac{dm}{dx}y(x) = m(x)Q(x). \tag{6}$$

The left hand side here is precisely what you get if you differentiate $m(x)y(x)$ using the product rule, so we get

$$\frac{dm(x)y(x)}{dx} = m(x)Q(x). \tag{7}$$

Integrate to get

$$m(x)y(x) = \int m(x)Q(x)\,dx \tag{8}$$

and thus

$$\boxed{y(x) = \frac{1}{m(x)}\int m(x)Q(x)\,dx.} \tag{9}$$

1.2. The Homogeneous and the Inhomogeneous Equations

Consider the linear differential equation (1)

$$A(x)\frac{dy}{dx} + B(x)y(x) = C(x)$$

again. This equation is called the *inhomogeneous equation.* The corresponding *homogeneous equation* is the equation you get by replacing the right hand side with 0. In other words, it's

$$A(x)\frac{dy}{dx} + B(x)y(x) = 0. \tag{10}$$

There are two basic important features of linear (differential) equations which are summarized in the following two theorems.

Theorem 1 (Superposition Principle)**.** *Let $y_1, y_2 : (a, b) \to \mathbb{R}$ be two solutions of the homogeneous equation* (10)*. Then for any two real numbers α and β the function $y_3(x) = \alpha y_1(x) + \beta y_2(x)$ is again a solution of the homogeneous differential equation* (10)*.*

Theorem 2 (About Particular Solutions)**.**

(i) *If $y_h : (a, b) \to \mathbb{R}$ is a solution of the homogeneous equation* (10)*, and if $y_p : (a, b) \to \mathbb{R}$ is a solution of the inhomogeneous equation* (1)*, then the sum $y(x) = y_p(x) + y_h(x)$ is also a solution of the inhomogeneous equation* (1)*.*

(ii) *If $y_1, y_2 : (a, b) \to \mathbb{R}$ are two solutions of the inhomogeneous equation* (1)*, then their difference $y_h(x) = y_1(x) - y_2(x)$ is a solution of the homogeneous equation.*

You should know the proofs of these theorems! They will be given in lecture.

The second theorem gives an alternative strategy for solving the inhomogeneous equation. Namely, first you find any old solution $y_p(x)$ of the inhomogeneous equation (it doesn't matter how, sometimes there's one obvious solution that stands out...). You call this solution a "particular solution." Next you solve the homogeneous equation, i.e. you find all solutions of (10). Theorem 2 then says that if $y_h(x)$ is your solution to the homogeneous equation, then the general solution to the inhomogeneous equation is given by

$$\boxed{y_{\text{inhom}}(x) = y_h(x) + y_p(x).} \tag{11}$$

2. SECOND ORDER EQUATIONS

2.1. Solving 2 × 2 systems of linear equations

From algebra you know how to solve a linear system of equations

$$\begin{cases} ax + by = p \\ cx + dy = q \end{cases} \tag{12}$$

in two unknowns x and y. For instance, you could multiply the first equation with d and the second with c, and subtract, with result $(ad - bc)x = pd - qc$. This gives you x. A similar trick will give you y. In the end the solution is given by

$$x = \frac{dp - bq}{ad - bc}, \quad y = \frac{aq - cp}{ad - bc}. \tag{13}$$

There is a special notation for the quantity $ad - bc$ which occurs in the denominator. It is called the *determinant* of the system, and is written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \stackrel{\text{def}}{=} ad - bc. \tag{14}$$

With this notation we can reformulate the above as follows

Theorem 3 ("Cramer's rule," the 2×2 case)**.** *If* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ *then the system of equations* (12) *has a solution for any given* $p, q \in \mathbb{R}$. *This solution is given by*

$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

2.2. The Wronskian and Abel's theorem

Consider the second order linear differential equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{15}$$

and its associated *homogeneous equation*

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0. \tag{16}$$

Here, and in the following all functions are assumed to be defined on some interval $x_1 < x < x_2$.

Definition. If y_1 and y_2 are solutions of the homogeneous equation, then their *Wronskian* is defined to be the function

$$W(x) \stackrel{\text{def}}{=} W(y_1, y_2; x) \stackrel{\text{def}}{=} y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Using the determinant notation we have therefore defined the Wronskian to be

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Theorem 4 (Niels Henrik Abel, (1802–1829))**.** *The Wronskian of two solutions of the linear homogeneous differential equation* (16) *satisfies*

$$\frac{dW}{dx} = -a(x)W(x).$$

Hence $W(x)$ is given by

$$W(x) = W(x_0)e^{\int_{x_0}^{x} a(x')dx'} \tag{17}$$

2.3. The Method of Variation of Constants

To solve the inhomogeneous equation (15) one can use the method of *Variation of Constants* (or "variation of parameters"). In this method one assumes that the solution y is given by

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x), \tag{18}$$

and that the functions c_1 and c_2 satisfy

$$y'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x), \tag{19}$$

Such functions c_1 and c_2 always exist, at least if the Wronskian of the two solutions y_1 and y_2 is nonzero (see problem 4). If this is so, then one has

$$\begin{aligned} y''(x) &= c_1'(x)y_1'(x) + c_2'(x)y_2'(x) && +c_1(x)y_1''(x) + c_2(x)y_2''(x) \\ a(x)y'(x) &= && a(x)c_1(x)y_1'(x) + a(x)c_2(x)y_2'(x) \\ b(x)y(x) &= && b(x)c_1(x)y_1(x) + b(x)c_2(x)y_2(x) \end{aligned}$$

Keep in mind that y_1 and y_2 both satisfy the homogeneous equation, and add vertically. You find that

$$f(x) = c_1'(x)y_1'(x) + c_2'(x)y_2'(x). \tag{20}$$

This gives us one equation for $c_1'(x)$ and $c_2'(x)$. To get a second equation we differentiate (18), applying the product rule, and combine the result with (19). One gets

$$0 = c_1'(x)y_1(x) + c_2'(x)y_2(x). \tag{21}$$

Equations (20) and (21) together form a system of two equations for the unknowns $c_1'(x)$ and $c_2'(x)$, namely

$$\begin{cases} y_1(x)c_1'(x) + y_2(x)c_2'(x) = 0 \\ y_1'(x)c_1'(x) + y_2'(x)c_2'(x) = f(x) \end{cases} \tag{22}$$

If the Wronskian $W(x) = y_1'(x)y_2(x) - y_1(x)y_2'(x)$ is nonzero, then one can solve this system for $c_1'(x)$ and $c_2'(x)$. Integrating $c_1'(x)$ and $c_2'(x)$ then gives $c_1(x)$ and $c_2(x)$, and from there you get the solution $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$.

When you work this out, you get

$$c_1'(x) = \frac{-y_2(x)f(x)}{W(x)}, \quad c_2'(x) = \frac{y_1(x)f(x)}{W(x)}, \tag{23}$$

where $W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the Wronskian of y_1 and y_2. Thus the solution of the inhomogeneous equation (15) is given by

$$y(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx. \tag{24}$$

Both indefinite integrals contain a constant, so the general solution we have found has two undetermined constants in it.

Besides giving us a method for solving the inhomogeneous equation, this computation also lets us prove a uniqueness theorem for the homogeneous equation.

Theorem 5. *Let $y_1, y_2 : (x_1, x_2) \to \mathbb{R}$ be two solutions of the homogeneous equation (16), for which the Wronskian $W(x)$ does not vanish. Then the general solution to the homogeneous equation (16) is*

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x). \tag{25}$$

Proof. The homogeneous equation is just a special case of the inhomogeneous equation where $f(x)$ hapens to vanish. So we can apply the method of Variation of Constants to get the general solution to the homogeneous equation by setting $f = 0$ in (24). The two integrals that appear in (24) now are:

$$\int \frac{y_2(x)f(x)}{W(x)}\,dx = \int 0\,dx = C_1, \quad \int \frac{y_1(x)f(x)}{W(x)}\,dx = \int 0\,dx = C_2.$$

Hence (24) says that the general solution is indeed given by (25). □

2.4. Linearity and the Superposition Principle

We abbreviate the lefthand side of the differential equation (15) by

$$\mathcal{L}[y] = y''(x) + a(x)y'(x) + b(x)y(x).$$

Thus the equations (15) and (16) can be written concisely as follows:

$$\text{inhomogeneous:} \quad \mathcal{L}[y] = f$$
$$\text{homogeneous:} \quad \mathcal{L}[y] = 0.$$

The expression $\mathcal{L}[y]$ is "linear in y," which, by definition, means that *for any two functions y_1 and y_2, and any two numbers c_1 and c_2 one has*

$$\mathcal{L}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{L}[y_1] + c_2 \mathcal{L}[y_2]. \tag{26}$$

Just as for 1st order equations one has a Superposition Principle.

Theorem 6 (Superposition Principle)**.**

(i) *If y_1 and y_2 are solutions of the homogeneous equation, then so is any linear combination $y = c_1 y_1 + c_2 y_2$ ($c_1, c_2 \in \mathbb{R}$).*

(ii) *If y_1 and y_2 are solutions of the inhomogeneous equations $\mathcal{L}[y_1] = f_1$ and $\mathcal{L}[y_2] = f_2$ respectively, then the linear combination $y = c_1 y_1 + c_2 y_2$ ($c_1, c_2 \in \mathbb{R}$ constants) satisfies $\mathcal{L}[y] = c_1 f_1 + c_2 f_2$.*

(iii) *If y_1 and y_2 are solutions to the same inhomogeneous equation, i.e. if $\mathcal{L}[y_1] = f$ and $\mathcal{L}[y_2] = f$, then their difference $y_h = y_1 - y_2$ satisfies the homogeneous equation: $\mathcal{L}[y_h] = 0$.*

2.5. Constant coefficient equations

There is no formula that gives you the general solution to the homogeneous equation for an arbitrary second order linear equation. But if the coefficients $a(x)$ and $b(x)$ are constant, such a formula does exist.

Consider the differential equation

$$y'' + py' + qy = 0, \tag{27}$$

where $p, q \in \mathbb{R}$ are constants. To solve this equation one looks for exponential functions which satisfy the equation. So set $y = e^{rx}$ for some constant r, and see if (27) holds:

$$y'' + py' + qy = r^2 e^{rx} + pre^{rx} + qe^{rx} = \left(r^2 + pr + q\right)e^{rx}.$$

Since $e^{rx} \neq 0$ no matter what r and x are (even if they are complex numbers!) we see that $y = e^{rx}$ is a solution of the homogeneous equation *if and only if* r satisfies the quadratic equation

$$r^2 + pr + q = 0. \tag{28}$$

There are now three cases:

$\boxed{p^2 - 4q > 0}$ In this case the characteristic equation has two real roots, r_1 and r_2, and we get two solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ of the homogeneous equation. it follows from the superposition principle that

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a solution of the homogeneous equation for any $c_1, c_2 \in \mathbb{R}$.

$\boxed{p^2 - 4q < 0}$ In this case the characteristic equation has two complex roots, which we write as

$$r_\pm = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\Omega,$$

where $\alpha = -p/2$ and $\Omega = \frac{1}{2}\sqrt{4q - p^2}$ are real numbers, and $\Omega > 0$. The solutions in exponential form are now

$$y_\pm(x) = e^{r_\pm x} = e^{(\alpha \pm i\Omega)x} = e^{\alpha x} e^{\pm i\Omega x} = e^{\alpha x}\big(\cos\Omega x \pm i \sin\Omega x\big)$$

These solutions are compex valued. To get real valued solutions one forms these linear combinations:

$$y_1(x) = \frac{1}{2}(y_+(x) + y_-(x)) = e^{\alpha x}\cos\Omega x$$
$$y_2(x) = \frac{1}{2i}(y_+(x) - y_-(x)) = e^{\alpha x}\sin\Omega x$$

Thus we find the following solutions for the homogeneous equation in this case:

$$y(x) = Ay_1(x) + By_2(x) = e^{\alpha x}\big(A\cos\Omega x + B\sin\Omega x\big).$$

$\boxed{p^2 - 4q = 0}$ In this last case the characteristic equation has one double root, namely $r = -p/2$. There is therefore only one exponential function $y(x) = e^{rx}$ which satisfies the equation. It turns out that in this case there is another solution which is not exponential, namely, xe^{rx}. So in this case we have the following solution to our constant coefficent equation (27),

$$y(x) = c_1e^{rx} + c_2xe^{rx} = \big(c_1 + c_2x\big)e^{rx}.$$

2.6. An Example

Problem: Solve the equation

$$y'' - y = f(x) \text{ (in class we had } f(x) = e^x\text{)}.$$

Solution: The homogeneous equation is $y'' - y = 0$, which is a constant coefficient equation. So you can find its solutions by trying an exponential $y = e^{rx}$. This leads to the characteristic equation $r^2 - 1 = 0$, whose solutions are $r = \pm 1$. We get two solutions

$$y_1(x) = e^x, \text{ and } y_2(x) = e^{-x}.$$

The Wronskian of these two solutions is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

Since the Wronskian does not vanish the solutions y_1 and y_2 are suitable for the method of Variation of Constants.

We write the unknown solution to our problem as

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) = c_1(x)e^x + c_2(x)e^{-x},$$

and require additionally that

$$y'(x) = c_1(x)y_1'(x) + c_2(x)y_2'(x) = c_1(x)e^x - c_2(x)e^{-x}.$$

This happens if and only if

$$0 = c_1'(x)e^x + c_2'(x)e^{-x}. \tag{29}$$

Substituting $y = c_1 e^x + c_2 e^{-x}$ in the differential equation, we get

$$\begin{aligned} f(x) &= y'' - y \\ &= c_1'(x)e^x + c_1(x)e^x + c_2'(x)e^{-x} - c_2(x)e^{-x} - \big(c_1(x)e^x + c_2(x)e^{-x}\big). \\ &= c_1'(x)e^x - c_2'(x)e^{-x}. \end{aligned}$$

Thus we have two equations for $c_1'(x)$ and $c_2'(x)$, namely

$$\begin{aligned} c_1'(x)e^x + c_2'(x)e^{-x} &= 0 \\ c_1'(x)e^x - c_2'(x)e^{-x} &= f(x). \end{aligned}$$

Solving these gives

$$c_1'(x) = \frac{1}{2}f(x)e^{-x}, \qquad c_2'(x) = -\frac{1}{2}f(x)e^x.$$

In the example in class we had $f(x) = e^x$, so

$$c_1'(x) = \frac{1}{2}, \qquad c_2'(x) = -\frac{1}{2}e^{2x}$$

and hence, by integration,

$$c_1(x) = \frac{x}{2} + C_1, \qquad c_2(x) = -\frac{1}{4}e^{2x} + C_2$$

where C_1 and C_2 are constants.

The general solution to our differential equation therefore is

$$\begin{aligned} y(x) &= c_1(x)e^x + c_2(x)e^{-x} \\ &= \frac{1}{2}xe^x - \frac{1}{4}e^{2x-x} + C_1e^x + C_2e^{-x} \\ &= \frac{1}{2}xe^x + \big(C_1 - \tfrac{1}{4}\big)e^x + C_2e^{-x}. \end{aligned}$$

You can simpify the appearance of this general solution by calling $A = C_1 - \frac{1}{4}$ ad $B = C_2$. You then get

$$y(x) = \frac{1}{2}xe^x + Ae^x + Be^{-x}.$$

where A and B are arbitrary constants.

2.7. Questions about the Theory

1 Derive the solutions in (13) (i.e. check those formulas.)

2 Solve

$$\begin{cases} x + iy = 2 + 6i \\ ix + y = -4 \end{cases} \quad \text{and} \quad \begin{cases} (2+i)x - 2y = 2 + 6i \\ -2x + (2-i)y = -4 \end{cases}$$

"by hand," and again using Cramer's rule.

3 Prove Abel's theorem: verify that the Wronskian really does satisfy $W'(x) = -a(x)W(x)$.

4 Suppose y_1 and y_2 are two solutions of the homogeneous equation (16) whose Wronskian does not vanish. *Show:* If y is an arbitrary differentiable function, then there always exist functions $c_1(x)$ and $c_2(x)$ such that (18) and (19) hold. (Can you write down a formula for c_1 and c_2 in terms of y, y', y_1, y_2, y_1' and y_2'?)

5 Which are the known functions, and which are the unknown functions in (22)?

6 Prove that (21) does indeed follow from the assumptions (18) and (19).

7 (i) State the definition of the statement "$\mathcal{L}[y]$ is linear in y."
(ii) Show that the expression $\mathcal{L}[y] = y''(x) + a(x)y'(x) + b(x)y(x)$ is indeed linear in y.

8 Consider the operator $\mathcal{M}$ defined by $\mathcal{M}[y] = \frac{dy}{dx} + y^2$.
(i) Compute $\mathcal{M}[y]$ when y is the function $y(x) = \sin x$. Do the same for $y = 2\sin x$.
(ii) Is $\mathcal{M}[y]$ linear in y?

9 Prove Theorem 6!!

2.8. Problems to practice the solution methods

10 Find the general solutions to the following diffeqs:

$$2y''(x) + 3\,y'(x) + y(x) = 0 \qquad y''(x) - 16y(x) = 0$$
$$y''(x) + Ay'(x) + y(x) = 0 \qquad y''(x) - y'(x) + Ay(x) = 0$$

where $A > 0$ is some constant.

11 Explain how you can use the Superposition Principle (Theorem 6) to find a particular and from there the general solution to the differential equations

$$y''(x) + \omega^2 y(x) = 1 + x + x^2$$
$$y''(x) + 2y'(x) - y(x) = x^2 + \sin(Ax)$$
$$6y''(x) + 5y'(x) + y(x) = e^{Ax}$$
$$y'' - 3y' + 2y = e^{Ax}$$

Here A and ω are positive constants.

12 Use Variation of Constants to find the general solution of the following equations:

$$y''(x) - y(x) = x$$
$$y'' + y = e^x + \sin x$$
$$y''(x) + 4y(x) = \sin Ax \qquad A > 0 \text{ is some constant.}$$
$$y''(x) - y'(x) = e^{i\omega x} \qquad \omega > 0 \text{ is a constant.}$$

13 Instead of choosing $y_1(x) = e^x$ and $y_2(x) = e^{-x}$ in the example of § 2.6, we could have chosen $y_1(x) = \sinh x$ and $y_2(x) = \cosh x$. Go through the same example with this choice of y_1 and y_2. Check that you get the same solutions!