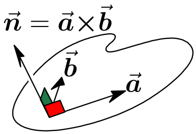

7.3. Finding the normal to a plane. If you know two vectors $\vec{a}$ and $\vec{b}$ which are parallel to a given plane $\mathcal{P}$ but not parallel to each other, then you can find a normal vector for the plane $\mathcal{P}$ by computing

$$\vec{n} = \vec{a}\times\vec{b}.$$

We have just seen that the vector $\vec{n}$ must be perpendicular to both $\vec{a}$ and $\vec{b}$, and hence[3] it is perpendicular to the plane $\mathcal{P}$.

This trick is especially useful when you have three points A, B and C, and you want to find the defining equation for the plane $\mathcal{P}$ through these points. We will assume that the three points do not all lie on one line, for otherwise there are many planes through A, B and C.

To find the defining equation we need one point on the plane (we have three of them), and a normal vector to the plane. A normal vector can be obtained by computing the cross product of two vectors parallel to the plane. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are both parallel to $\mathcal{P}$, the vector $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC}$ is such a normal vector.

Thus the defining equation for the plane through three given points A, B and C is

$$\vec{n}\cdot(\vec{x}-\vec{a}) = 0, \quad \text{with} \quad \vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = (\vec{b}-\vec{a})\times(\vec{c}-\vec{a}).$$

7.4. Example. Find the defining equation of the plane $\mathcal{P}$ through the points $A(2,-1,0)$, $B(2,1,-1)$ and $C(-1,1,1)$. Find the intersections of $\mathcal{P}$ with the three coordinate axes, and find the distance from the origin to $\mathcal{P}$.

Solution: We have

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

so that

$$\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$$

is a normal to the plane. The defining equation for $\mathcal{P}$ is therefore

$$0 = \vec{n}\cdot(\vec{x}-\vec{a}) = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 2 \\ x_2 + 1 \\ x_3 - 0 \end{pmatrix}$$

i.e.

$$4x_1 + 3x_2 + 6x_3 - 5 = 0.$$

The plane intersects the x_1 axis when $x_2 = x_3 = 0$ and hence $4x_1 - 5 = 0$, i.e. in the point $(\frac{5}{4}, 0, 0)$. The intersections with the other two axes are $(0, \frac{5}{3}, 0)$ and $(0, 0, \frac{5}{6})$.