234 Review Sheet 2 Solutions

1. Classify the following quadratic forms:

   (a)

$$\begin{aligned} x^2 - 4xy + 5y^2 &= x^2 - 4xy + (2y)^2 - (2y)^2 + 5y^2 \\ &= (x - 2y)^2 + y^2 \end{aligned}$$

   Is of the form $aX^2 + bY^2$ for $a, b > 0$. This is positive definite.

   Using the discriminant $4AC - B^2 = 20 - 16 = 4 > 0$. Also $A = 1 > 0$. Therefore it is positive definite.

   (b)

$$\begin{aligned} 3x^2 + xy - 2y^2 &= 3(x^2 + \frac{1}{3}xy - \frac{2}{3}y^2) \\ &= 3(x^2 + \frac{1}{3}xy + (\frac{1}{6}y)^2 - (\frac{1}{6}y)^2 - \frac{2}{3}y^2) \\ &= 3\left((x + \frac{1}{6}y)^2 - \frac{23}{36}y^2\right) \\ &= 3(x + \frac{1}{6}y)^2 - \frac{23}{12}y^2 \end{aligned}$$

   Is of the form $aX^2 - bY^2$ for $a, b > 0$. This is indefinite.

   Using the discriminant $4AC - B^2 = -24 - 1 = -25 < 0$. Therefore it is indefinite.

   (c)

$$\begin{aligned} x^2 + 10xy + y^2 &= x^2 + 10xy + (5y)^2 - (5y)^2 + y^2 \\ &= (x + 5y)^2 - 24y^2 \end{aligned}$$

   Is of the form $aX^2 - bY^2$ for $a, b > 0$. This is indefinite.

   Using the discriminant $4AC - B^2 = 4 - 100 = -96 < 0$. Therefore it is indefinite.

   (d)

$$4x^2 - xy = x(4x - y)$$

   factors. This is indefinite.

   Using the discriminant $4AC - B^2 = 0 - 1 = -1 < 0$. Therefore it is indefinite.

(e)

$$6xy = (6x)(y)$$

factors. This is indefinite.

Using the discriminant $4AC - B^2 = 0 - 36 = -36 < 0$. Therefore it is indefinite.

(f)

$$(x - y)(x - 2y) = x^2 - 3xy + 2y^2$$

factors. This is indefinite.

Using the discriminant $4AC - B^2 = 8 - 9 = -1 < 0$. Therefore it is indefinite.

2. Consider the following functions $f(x, y)$, do each of the following *in order*

- Find the level set $f(x, y) = 0$
- Using the implicit function theorem find all points on the level set such that, around those points, the graph of the level set is the graph of a function $y = g(x)$.
- At every such point (x_0, y_0), find $g'(x)$ (NOTE $g(x)$ may be different at different points. There may not be an "explicit formula" for $g(x)$, use the implicit function theorem for this question)

(a) $f(x, y) = x + y$

- The level set is $x + y = 0$ the line $y = -x$.
- $f_y = 1 \neq 0$ so by the implicit function theorem the level set can be represented by a function $y = g(x)$ everywhere (in fact $y = -x$ works)
- By the implicit function theorem, $g'(x) = \frac{-f_x}{f_y} = -1$

(b) $f(x, y) = x^2 + y^2$

- The level set is $x^2 + y^2 = 0$ the point $(0, 0)$.
- $f_y = 2y$ which is zero on the entire level set, so there are no such points.
- Not applicable.

This is a special example, as the level set is just a point and not a curve. However, in every neighborhood of the point (0,0), the graph of *any* continuous function $y = g(x)$ is a curve and not a point.

(c) $f(x, y) = x^2 - y^2$

- The level set is $x^2 - y^2 = 0$, the set of points the line $y = x$ or $y = -x$.
- $f_y = -2y$ which is zero when $y = 0$. So It is all points such that $y = \pm x \neq 0$
- By the Implicit function theorem $g'(x) = \frac{-f_x}{f_y} = \frac{-2x}{-2y} = \frac{x}{y}$.

3. Consider the following functions $f(x, y, z)$, do each of the following *in order*

- Find the level set $f(x, y, z) = 5$

- Using the implicit function theorem find all points on the level set such that, around those points, the graph of the level set is the graph of a function $y = g(x,z)$.
- At every such point (x_0, y_0, z_0), find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial z}$ (NOTE $g(x,z)$ may be different at different points. There may not be an "explicit formula" for $g(x)$, use the implicit function theorem for this question)
- Find an equation for the tangent plane to the level set $f(x,y,z) = 5$ at the point p.

(a) $f(x,y,z) = x^2 + 2y^2 + 3z^2$

- The level set is an ellipsoid $x^2 + 2y^2 + 3z^2 = 5$
- $f_y = 4y$ which is zero when $y = 0$. So it can be represented as a function $y = g(x,z)$ in a neighborhood of every point in the level set not on the ellipse $x^2 + z^2 = 5$
- By the implicit function theorem

$$\frac{\partial g}{\partial x} = \frac{-f_x}{f_y} = \frac{-2x}{4y} = \frac{-x}{2y}$$
$$\frac{\partial g}{\partial z} = \frac{-f_z}{f_y} = \frac{-6z}{4y} = \frac{-3z}{2y}$$

- Using the linear approximationg formula with $p = (0,1,1)$, we get

$$y = 1 + \frac{-(0)}{2(1)}(x-0) + \frac{-3(1)}{2(1)}(z-1)$$
$$y = 1 - \frac{3}{2}(z-1)$$

(b) $f(x,y,z) = \ln(xyz) + y$

- The level set is the graph of $z = \frac{e^{5-y}}{xy}$
- $f_y = \frac{1}{y} + 1$ which is zero when $y = -1$. So it can be represented as a function $y = g(x,z)$ in a neighborhood of every point in the level set not on the curve $z = \frac{-e^6}{x}$
- By the implicit function theorem

$$\frac{\partial g}{\partial x} = \frac{-f_x}{f_y} = \frac{1/x}{1/y+1} = \frac{y}{x(y+1)}$$
$$\frac{\partial g}{\partial z} = \frac{-f_z}{f_y} = \frac{1/z}{1/y+1} = \frac{y}{z(y+1)}$$

- Using the linear approximationg formula with $p = (1/5, 5, 1)$, we get

$$y = 5 + \frac{(5)}{(1/5)(5+1)}(x - 1/5) + \frac{(5)}{(1)(5+1)}(z-1)$$
$$y = 5 + \frac{25}{6}(x - \frac{1}{5}) + \frac{5}{6}(z-1)$$

(c) $f(x,y,z) = e^x + e^y + e^z$

- The level set is the graph of $z = \ln(5 - e^x - e^y)$
- $f_y = e^y$ which is never zero. So it can be represented as a function $y = g(x,z)$ everywhere
- By the implicit function theorem

$$\frac{\partial g}{\partial x} = \frac{-f_x}{f_y} = \frac{e^x}{e^y} = e^{x-y}$$
$$\frac{\partial g}{\partial z} = \frac{-f_z}{f_y} = \frac{e^z}{e^y} = e^{z-y}$$

- Using the linear approximationg formula with $p = (0, \ln 2, \ln 2)$, we get

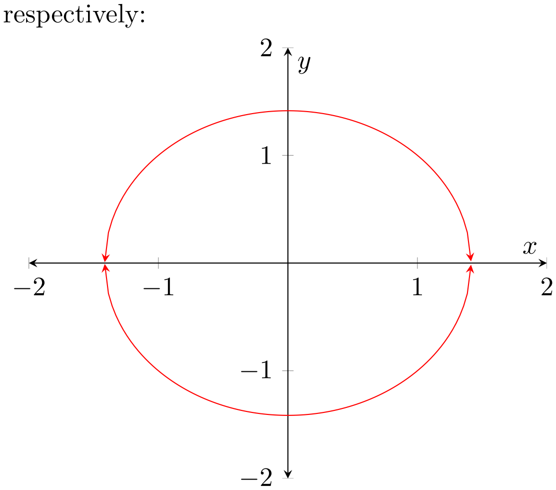
$$y = \ln 2 + e^{0-\ln 2}(x-0) + e^{\ln 2 - \ln 2}(z - \ln 2)$$
$$y = \frac{1}{2}x + z$$

4. For each of the specified functions $f(x,y)$ and points p do the following:

- Convert the function to a function $g(r,\theta)$ in polar coordinates
- Find the partal derivatives of f with repect to x, y, r, θ.
- Find the tangent plane to the surface $z = f(x,y)$ at the point p.
- Find and draw the level sets of $z = f(x,y)$ at $z = 0, 2, -1$.
- Suppose $x(t) = t + \ln(t)$ and $y(t) = \cos(t)$, and find $\frac{df}{dt}$.

(a) $f(x,y) = x^2 + y^2$, $p = (3, 2, 13)$

- $g(r,\theta) = f(r\cos(\theta), r\sin(\theta)) = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$
- $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial r} = 2r$, $\frac{\partial f}{\partial \theta} = 0$
- The tangent plane is given by the equation $z = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$. This gives us $z = 13 + 6x - 18 + 4y - 8 = 6x + 4y - 5$.
- Let blue, red, and green be the colors of the level sets at 0, 2, and -1 respectively:

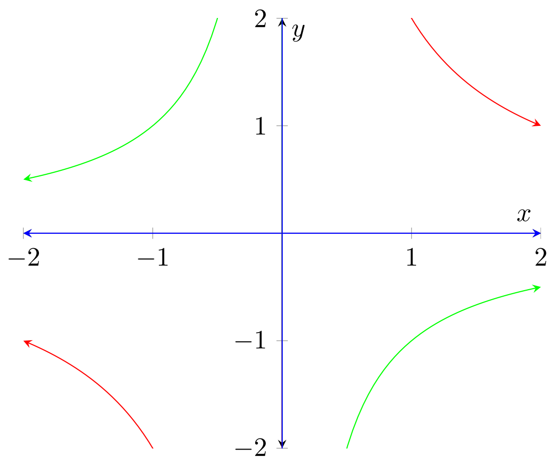


The level set at 0 is just the origin, hard to see in the picture. The level set at -1 is empty.

- $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 2(t + \ln(t))\left(1 + \frac{1}{t}\right) - 2\cos(t)\sin(t)$

(b) $f(x, y) = xy$, $p = (1, 1, 1)$

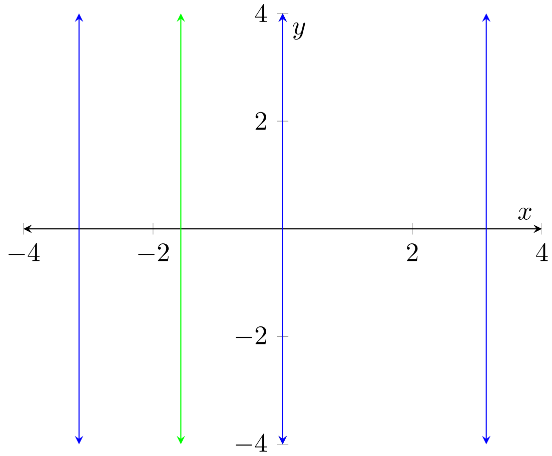
- $g(r, \theta) = f(r\cos(\theta), r\sin(\theta)) = r^2\cos(\theta)\sin(\theta)$
- $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial f}{\partial r} = 2r\cos(\theta)\sin(\theta)$, $\frac{\partial f}{\partial \theta} = r^2(\cos^2(\theta) - \sin(\theta)^2)$
- The tangent plane is given by the equation $z = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$. This gives us $z = 1 + x - 1 + y - 1 = x + y - 1$.
- Let blue, red, and green be the colors of the level sets at 0, 2, and -1 respectively:



- $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \cos(t)\left(1 + \frac{1}{t}\right) - (t + \ln(t))\sin(t)$

(c) $f(x, y) = \sin(x)$, $p = (0, 0, 0)$

- $g(r, \theta) = f(r\cos(\theta), r\sin(\theta)) = \sin(r\cos(\theta))$
- $\frac{\partial f}{\partial x} = \cos(\theta)$, $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial r} = \cos(r\cos(\theta))\cos(\theta)$, $\frac{\partial f}{\partial \theta} = \cos(r\cos(\theta))(-r\sin(\theta))$
- The tangent plane is given by the equation $z = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0)$. This gives us $z = x$.
- Let blue, red, and green be the colors of the level sets at 0, 2, and -1 respectively:

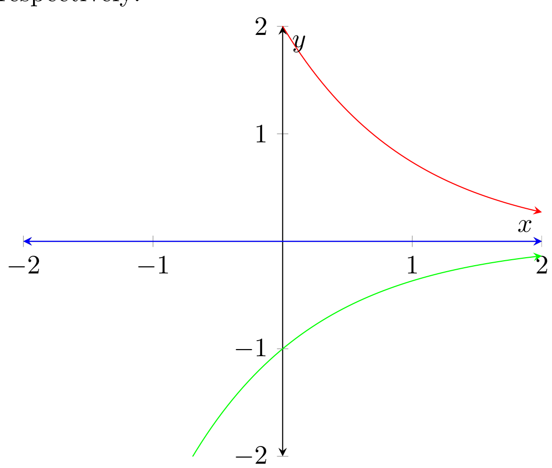


Noting that $\sin(x) = 2$ has no solutions.

- $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = \cos(t + \ln(t))\left(1 + \dfrac{1}{t}\right)$

(d) $f(x, y) = e^x y$, $p = (1, 1, e)$

- $g(r, \theta) = f(r\cos(\theta), r\sin(\theta)) = e^{r\cos(\theta)} r\sin(\theta)$
- $\dfrac{\partial f}{\partial x} = e^x y$, $\dfrac{\partial f}{\partial y} = e^x$, $\dfrac{\partial f}{\partial r} = e^{r\cos(\theta)} r\cos(\theta)\sin(\theta) + e^{r\cos(\theta)}\sin(\theta)$, $\dfrac{\partial f}{\partial \theta} = -e^{r\cos(\theta)} r^2 \sin^(\theta) + e^{r\cos(\theta)} r\cos(\theta)$
- The tangent plane is given by the equation $z = f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$. This gives us $z = e + e(x - 1) + e(y - 1) = e(x + y - 1)$.
- Let blue, red, and green be the colors of the level sets at 0, 2, and -1 respectively:



- $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} = e^{t+\ln(t)}\cos(t)\left(1 + \dfrac{1}{t}\right) - e^{t+\ln(t)}\sin(t)$

5. Suppose you are in a spaceship nearing a black hole. The gravitational force of the

black hole is given by

$$G(x,y,z) = \frac{1}{x^2+y^2+z^2-1}$$

For $x^2+y^2+z^2 > 1$. All points such that $x^2+y^2+z^2 \leq 1$ are inside the "event horizon" of the black hole and can never escape. To avoid getting turned into spaghetti, you want to move away from the black hole as quickly as possible in a way that the gravitational force decreases the most. If your spaceship is sitting at the point (3,2,3), what dircetion should you pilot your spaceship in?

You want to pilot your ship in the direction which G has the largest rate of *decrease*. This is $-\vec{\nabla}G$.

$$-\vec{\nabla}G = \frac{1}{(x^2+y^2+z^2-1)^2}\begin{pmatrix}2x\\2y\\2z\end{pmatrix}$$

So you want to pilot your spaceship in the direction $-\vec{\nabla}G(3,2,3) = \frac{1}{21^2}\begin{pmatrix}6\\4\\6\end{pmatrix}$.