Math 234, 1st midterm with solutions, fall 2017, lecture 001

1. (44%) Consider the points $A(1,1,2)$, $B(-1,-1,2)$, $C(1,-1,-1)$, and $D(-1,1,-1)$.

(a) Find the distance from the point A to the plane through the points B, C, and D.

First find the equation for the plane through B, C, D: a normal vector for the plane is

$$\vec{\boldsymbol{n}} = \overrightarrow{BC} \times \overrightarrow{BD} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix}.$$

We also need a point on the plane, we choose B (could also have chosen C or D). The equation for the plane then is $\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}} = \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{b}}$, i.e.

$$6x_1 + 6x_2 + 4x_3 = -6 - 6 + 8 = -4.$$

The distance from the point A to the plane is given by

$$\pm d = \frac{\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}})}{\|\vec{\boldsymbol{n}}\|} = \frac{1}{\sqrt{6^2+6^2+4^2}} \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1-(-1) \\ 1-(-1) \\ 2-2 \end{pmatrix} = \frac{10}{\sqrt{6^2+6^2+4^2}} = \frac{10}{\sqrt{88}}.$$

(b) Find the cosine of the angle between the plane through BCD and the xy-plane. (Hint: what is a normal to the xy-plane?)

The angle between two planes is the angle between their normals. A normal to BCD is the vector $\vec{\boldsymbol{n}}$ from part (a) of this problem. A normal to the xy-plane is $\vec{\boldsymbol{k}} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$.

The cosine of the angle between the two planes is

$$\cos\angle(BCD, xy\text{-plane}) = \cos\angle(\vec{\boldsymbol{n}}, \vec{\boldsymbol{k}}) = \frac{\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{k}}}{\|\vec{\boldsymbol{n}}\| \, \|\vec{\boldsymbol{k}}\|} = \frac{4}{\sqrt{88} \cdot 1} = \frac{4}{\sqrt{88}}.$$

(c) Do the points A and $E(-1,-1,-1)$ lie on the same side of the plane BCD?

The sign of $\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{x}} - \vec{\boldsymbol{b}})$ tells us if the point with position vector $\vec{\boldsymbol{x}}$ lies on the side of the plane BCD that $\vec{\boldsymbol{n}}$ points to. So to see if A and E are on the same side, we compare the signs of $\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}})$ and $\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{e}} - \vec{\boldsymbol{b}})$.

We have already computed $\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}) = +10$; for the new point E we have

$$\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{e}} - \vec{\boldsymbol{b}}) = \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} (-1)-(-1) \\ (-1)-(-1) \\ (-1)-2 \end{pmatrix} = 4 \cdot (-3) = -12.$$

So $\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{a}} - \vec{\boldsymbol{b}}) = +10$ and $\vec{\boldsymbol{n}} \cdot (\vec{\boldsymbol{e}} - \vec{\boldsymbol{b}}) = -12$ have opposite signs, which means that A and E lie on opposite sides of the plane BCD.

(d) Find a parametric representation for the line through A and C, and determine where this line intersects the three coordinate planes (i.e. the xy-plane, the yz-plane, and the xz-plane).

The direction of the line is given by the vector $\overrightarrow{AC} = \vec{c} - \vec{a} = \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}$.

A parametrization of the line is then given by

$$\vec{x} = \vec{a} + t\overrightarrow{AC} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-2t \\ 1-2t \end{pmatrix}.$$

We see that $\vec{x}$ lies on the xy plane when its third component vanishes, i.e. when $1 - 2t = 0$. This happens for $t = \frac{1}{2}$. The point where the line intersects the xy-plane is therefore $(1, 0, 0)$.

To see where the line intersects the xz-plane, we check when the second component of $\vec{x}$ vanishes. The second component is the same as the third, namely $1 - 2t$, so we again get $t = \frac{1}{2}$, and the intersection point is $(1, 0, 0)$.

Finally, the first component of $\vec{x}$ is 1 for all values of t, so it never vanishes. This line does not intersect the yz-plane.

2. (44%) The vector function $\vec{x}(t) = \begin{pmatrix} \tan t \\ t \end{pmatrix}$ describes the motion of a point $X(t)$ in the plane.

(a) Compute the velocity, speed, and acceleration of the point at any time t. The velocity and acceleration are vectors. They are the first and second derivatives of the position vector, so

$$\vec{v}(t) = \vec{x}'(t) = \begin{pmatrix} 1/\cos^2 t \\ 1 \end{pmatrix}, \qquad \vec{a}(t) = \vec{x}''(t) = \begin{pmatrix} 2\sin t/\cos^3 t \\ 0 \end{pmatrix}.$$

The speed is the length of the velocity vector, i.e.

$$v(t) = \|\vec{v}(t)\| = \sqrt{\frac{1}{\cos^4 t} + 1}.$$

(b) At which moment(s) in time t are the point's velocity and acceleration perpendicular? Velocity and acceleration are perpendicular if their dot-product vanishes. We compute the dot-product:

$$\vec{v}(t) \cdot \vec{a}(t) = \begin{pmatrix} 1/\cos^2 t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2\sin t/\cos^3 t \\ 0 \end{pmatrix} = 2\frac{\sin t}{\cos^5 t}.$$

This dot-product equals zero if $\sin t = 0$, i.e. when $t = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

(c) Where does the tangent line to the curve at the point $X(t)$ intersect the x axis? A parametrization of the tangent at the point $X(t)$ is given by

$$\vec{y} = \vec{x}(t) + s\vec{v}(t).$$

Here t determines at which point on the curve we are finding the tangent, and s is the parameter along the tangent line. So we get

$$\vec{\boldsymbol{y}} = \begin{pmatrix} \tan t \\ t \end{pmatrix} + s \begin{pmatrix} 1/\cos^2 t \\ 1 \end{pmatrix} = \begin{pmatrix} \tan t + s/\cos^2 t \\ t + s \end{pmatrix}.$$

$\vec{\boldsymbol{y}}$ will be the position vector of a point on the x-axis if the second component of $\vec{\boldsymbol{y}}$ vanishes, i.e. if $s = -t$. Therefore the intersection point with the x-axis has position vector

$$\vec{\boldsymbol{y}} = \begin{pmatrix} \tan t - t/\cos^2 t \\ 0 \end{pmatrix}.$$

(d) Find an integral for the length of the part of the curve for which $0 \le t \le \pi/4$ (no need to compute the integral).

The length of the segment is the integral of "*speed* d(*time*)," i.e.

$$\text{length} = \int_0^{\pi/4} v(t)dt = \int_0^{\pi/4} \sqrt{\frac{1}{\cos^4 t} + 1}\, dt.$$

Note: the original question asked for the length of the segment from $t = 0$ to $t = \pi$. This segment is infinitely long as it includes the t-value $t = \pi/2$, at which $\tan t \to \infty$.

3. (12%) Suppose a point $X(t)$ is moving through space, and suppose its position vector at time t is $\vec{\boldsymbol{x}}(t)$.

Assume that the position vector $\vec{\boldsymbol{x}}(t)$ and the acceleration $\vec{\boldsymbol{a}}(t) = \vec{\boldsymbol{x}}''(t)$ are parallel at every instant in time t.

Explain why the vector $\vec{\boldsymbol{x}}(t) \times \vec{\boldsymbol{x}}'(t)$ does not depend on time by simplifying $\frac{d}{dt}\left(\vec{\boldsymbol{x}}(t) \times \vec{\boldsymbol{x}}'(t)\right)$.

By the product rule for cross products of vector functions we have

$$\frac{d\vec{\boldsymbol{x}}(t) \times \vec{\boldsymbol{x}}'(t)}{dt} = \frac{d\vec{\boldsymbol{x}}}{dt} \times \vec{\boldsymbol{x}}'(t) + \vec{\boldsymbol{x}}(t) \times \frac{d\vec{\boldsymbol{x}}'}{dt} = \vec{\boldsymbol{x}}'(t) \times \vec{\boldsymbol{x}}'(t) + \vec{\boldsymbol{x}}(t) \times \vec{\boldsymbol{x}}''(t).$$

The first term vanishes because the cross product of any vector with itself vanishes. The second term also vanishes because $\vec{\boldsymbol{x}}(t)$ and $\vec{\boldsymbol{x}}''(t)$ are given to be parallel, and the cross product of two vectors that are parallel always vanishes.

Therefore the derivative of $\vec{\boldsymbol{x}} \times \vec{\boldsymbol{x}}'$ vanishes at every moment in time. This implies that $\vec{\boldsymbol{x}} \times \vec{\boldsymbol{x}}'$ does not depend on time, i.e. it is constant.