MATH 234–PREVIOUS FINALS

1. Final of 2013

(1) Let $f(x, y) = xe^{x-2y}$.
  (a) Find the linear approximation of $f(x, y)$ near $x = 2, y = 1$.
  (b) Use the linear approximation to approximate $f(1.99, 1.02)$.

(2) Suppose we are given a function $f(x, y)$. Define $g(u, v) = f(u \ln v, u + v)$. In the following questions your answer may contain the variables u and v as well the partial derivatives of f with respect to x and/or y.
  (a) Compute $\frac{\partial g}{\partial u}$.
  (b) Compute $\frac{\partial^2 g}{\partial u \partial v}$.

(3) (a) Find all critical points of the function $f(x, y) = x^2 - 3y^2 + y^3$;
  (b) Use the second derivative test to tell which of the critical points are local maxima, minima, or saddle points. For any saddle points find the tangent lines to the level set.

(4) Use the method of Lagrange multipliers to find the largest and smallest values that xy^2 can have if $x^2 + y^2 = 3$.

(5) Let V be the volume under the graph of

$$z = \frac{1}{(x + y)^2}$$

above the rectangle in which $3 \le x \le 6$ and $0 \le y \le 2$.
  (a) Which integral do you have to compute to find V?
  (b) Compute V.

(6) Consider the integral

$$I = \iint_{\mathcal{R}} f(x, y)\, dA = \int_1^5 \int_{(x-1)/2}^{\sqrt{x-1}} f(x, y)\, dy\, dx$$

  (a) Draw the domain $\mathcal{R}$
  (b) Find the integration bounds that appear when you rewrite the integral in the form

$$I = \int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots} f(x, y)\, dx\, dy$$

(7) Let $\mathcal{R}$ be the three dimensional region given by $x \geq 0$, $y \geq x$, and $x^2 + y^2 \leq 4$, and $0 \leq z \leq x^2 + y^2$.
   (a) Describe $\mathcal{R}$ in cylindrical coordinates (r, θ, z).
   (b) Compute the average of z in $\mathcal{R}$.

(8) Let $\mathcal{R}$ be the three dimensional region inside the sphere $x^2 + y^2 + z^2 \leq 9$ that is given by $x \geq 0$, $y \geq 0$, $z \geq 0$, and $y \geq x$.
   (a) Make a drawing that describes the spherical coordinates (ρ, ϕ, θ) of a point and describe the region $\mathcal{R}$ in spherical coordinates.
   (b) Write the integral

$$I = \iiint_{\mathcal{R}} x \, dV$$

   in terms of spherical coordinates. Do not compute the integral.

(9) Let A, B, and C be the three points $A(0,0)$, $B(1,1)$, $C(2,0)$.
   Let $\vec{\boldsymbol{F}}$ be the vectorfield $\vec{\boldsymbol{F}}(x,y) = \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$.
   (a) Compute $\int_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}}$ where $\mathcal{C}$ is the curve that consists of the line segment AB followed by the line segment BC (i.e. $\mathcal{C}$ is the top of the triangle ABC, oriented from left to right.)
   (b) Compute $\int_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}} \, ds$, where $\vec{\boldsymbol{N}}$ is the unit normal to $\mathcal{C}$ that points upward.
   (c) Use Green's theorem to compute the line integral

$$I = \int_{\mathcal{T}} \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{x}}$$

   where $\mathcal{T}$ is the triangle ABC with counter-clockwise orientation, and $\vec{\boldsymbol{G}}$ is the vector field

$$\vec{\boldsymbol{G}}(x,y) = \begin{pmatrix}0\\x\end{pmatrix}.$$

2. Final of 2015

(1) Suppose we are given one function $f(x,y)$ and then define a new function by setting

$$g(u,v) = f\big(\frac{u}{v}, uv\big).$$

In the following two questions your answer may contain the variables u and v as well the partial derivatives of f with respect to x and/or y.
   (a) Compute $\dfrac{\partial g}{\partial v}$.
   (b) Compute $\dfrac{\partial^2 g}{\partial u \partial v}$.

(2) Consider the function the function $f(x,y) = 6xy - x^2 - y^3$.
   (a) Find all critical points of f.
   (b) Use the second derivative test to tell which of the critical points of f are local maxima, minima, or saddle points. For any saddle points find the tangent lines to the level set.

(3) Let V_L be the volume of the region under the graph of

$$z = 15\sqrt{x+y}$$

and above the triangle given by $x + y \leq L$ and $x, y \geq 0$. Here $L > 0$ is a constant.

(a) Which integral do you have to compute to find V_L? Include the integration bounds in your answer.

(b) Compute V_L.

(4) Consider the integral

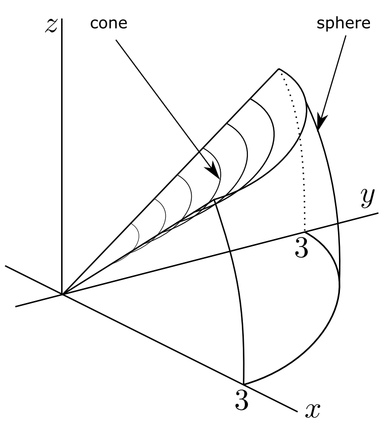
$$I = \iint_{\mathcal{R}} f(x,y)\,dA = \int_{x=0}^{2} \int_{y=0}^{4x-x^2} f(x,y)\,dy\,dx$$

(a) Draw the domain $\mathcal{R}$

(b) Find the integration bounds that appear when you rewrite the integral in the form

$$I = \int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots} f(x,y)\,dx\,dy$$

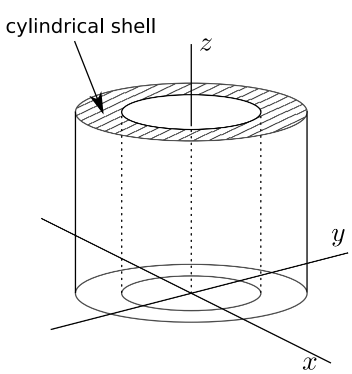
(5) Compute the volume of the three dimensional region $\mathcal{R}$ in the first octant, under the cone $z = \sqrt{x^2+y^2}$ and inside the sphere with radius 3.



Your solution should address these points:

- Which coordinates will you use?
- Describe the region $\mathcal{R}$ in your chosen coordinates,
- write the integral in the coordinates you chose,
- compute the integral.

(6) Compute the moment of inertia of the region $\mathcal{R}$, i.e. compute $\iiint_{\mathcal{R}} (x^2+y^2)\,dV$, where $\mathcal{R}$ is the cylindrical shell in the drawing. The height of the shell is H, its inner radius is a, and its outer radius is b.



Your solution should address these points:

- Which coordinates will you use?
- Describe the region $\mathcal{R}$ in your chosen coordinates,
- write the integral in the coordinates you chose,
- compute the integral.

(7) (a) Define the line integral

$$I = \int_{\mathcal{C}} x^2 dx - y^3 dy$$

where $\mathcal{C}$ is the circle with radius 5, centered at the origin, and with counter-clockwise orientation. Find a vector field $\vec{\boldsymbol{F}}$ such that

$$I = \int_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}}.$$

(b) Is $\vec{\boldsymbol{F}}$ (from part (a)) the gradient of some function?

(c) Compute the line integral I from part (a) and **explain your computation.**

(d) The velocity vector field of a flowing gas is given by $\vec{\boldsymbol{v}}(x, y) = \begin{pmatrix} -x \\ 1 \end{pmatrix}$.

Compute the flux of $\vec{\boldsymbol{v}}$ across the circle $\mathcal{C}$ from part (a), i.e. compute $\int_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{N}} \, ds$ where $\vec{\boldsymbol{N}}$ is the outward pointing normal to $\mathcal{C}$.