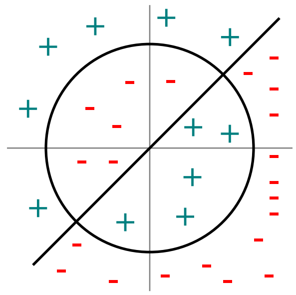
Midterm 3–solutions

(1) **(a)** (7%) Draw the zero set of the function $f(x, y) = (y - x)(x^2 + y^2 - 4)$, and use your drawing to predict as many critical points of f as you can without computing the derivatives of f. Explain your answer.



The zero set of the function is the union of the line $y = x$ and the circle with radius 2 and center at the origin.

The two points where the line intersects the circle are critical points because $\vec{\nabla} f$ is always orthogonal to any curve in the zero set.

The function must have a maximum in region below the line and inside the circle. The maximum cannot be on the boundary of that region, so it is an interior maximum. It must therefore be a critical point. Therefore the function has at least one critical point in the half of the disc below the line $y = x$. Note that there could be more than one critical point in this region. All we know is that there is at least one such critical point.

The function must also have a minimum in the region inside the circle above the line, and we conclude that f has at least one more critical point in the region above the line, inside the circle.

In total we know that $f(x, y)$ has at least four critical points.

(b) (18%) Find all the critical points of $f(x, y) = x^2 - 4xy + y^2 - 6x - 12y$ and use the second derivative test to decide which of these are local minima, local maxima, or saddle points. At saddle points compute the two tangent lines to the level set.

The derivatives are $f_x(x, y) = 2x - 4y - 6$, $f_y(x, y) = -4x + 2y - 12$.

Solving $f_x = 0$, $f_y = 0$, leads to $x = -5$, $y = -4$ (so there is only one critical point).

The second order Taylor expansion at the critical point is

$$f(-5 + \Delta x, -4 + \Delta y) = f(-5, -4) + (\Delta x)^2 - 4\Delta x\Delta y + (\Delta y)^2 + \text{error terms}.$$

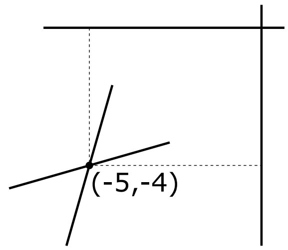
Completing the square in the second order terms we get

$$\big(\Delta x - 2\Delta y\big)^2 - 3(\Delta y)^2 = \big(\Delta x - (2 - \sqrt{3})\Delta y\big)\big(\Delta x - (2 + \sqrt{3})\Delta y\big).$$

This form is indefinite and therefore the critical point is a saddle point. The two tangents to the level set at the critical point are given by

$$\Delta x = (2 - \sqrt{3})\Delta y, \text{ and } \Delta x = (2 + \sqrt{3})\Delta y,$$

where $\Delta x = x - (-5) = x + 5$ and $\Delta y = y - (-4) = y + 4$.



The two tangents to the level set through the critical point

(2) **(a)** (20%) Where does the quantity $x + 4y$ attain its **largest** value provided (x, y) must satisfy $x^2 + 4y^2 = 20$? Explain your answer (which equations will you solve)?

This is a constrained optimization problem where we want to find the maximum of $f(x, y) = x + 4y$, subject to the constraint $g(x, y) = x^2 + 4y^2 = 20$.

The maximum occurs either at solutions of

$$\vec{\nabla} g = \vec{0}, \quad g(x, y) = 20 \qquad \text{(the exceptional case)}$$

or

$$\vec{\nabla} f = \lambda \vec{\nabla} g, \quad g = 20 \qquad \text{(Lagrange multiplier case)}$$

Exceptional case. We have $g_x = 2x$, $g_y = 4y$, so the equations $\vec{\nabla} g = \vec{0}$ boil down to $2x = 0$, $4y = 0$, and the only solution is $x = y = 0$. But this solution does not satisfy the constraint $g = 20$, so here the exceptional case does not yield solutions, and the maximum must be a solution of the Lagrange equations.

Lagrange equations. We must solve $\vec{\nabla} f = \lambda \vec{\nabla} g$, $g = 20$. These three equations are

$$1 = 2\lambda x, \quad 4 = 8\lambda y, \quad x^2 + 4y^2 = 20.$$

The first equation $1 = 2\lambda x$ implies that $x \neq 0$ (because $x = 0$ would give $2\lambda x = 0$). Therefore $\lambda = 1/(2x)$. The second equation implies $y \neq 0$ and $\lambda = 1/(2y)$. Hence $1/(2x) = 1/(2y)$, and thus $x = y$. Applying this to the constraint $g = 20$ we get $x^2 + 4x^2 = 20$, i.e. $5x^2 = 20$, so $x = \pm\sqrt{20/4} = \pm 2$.

We get two solutions: $(x, y) = (2, 2)$ with $\lambda = -1/4$, and $(x, y) = (-2, -2)$ with $\lambda = -1/4$.

Of these two solutions we have $f(2, 2) = 2 + 4 \times 2 = 10$ and $f(-2, -2) = -10$, so that the maximum is attained at $(2, 2)$, while the minimum is at $(-2, -2)$.

(b) (5%) Almost the same question as above: Where does the quantity $x + 4y$ attain its **largest** value provided (x, y) must satisfy $x^2 + 4y^2 = 0$? Explain your answer (which equations will you solve)?

The constraint here is $x^2 + 4y^2 = 0$. There is only one point that satisfies this constraint, namely, $(x, y) = (0, 0)$. The largest value that $x + 4y$ therefore can have subject to this constraint is $f(0, 0) = 0$.

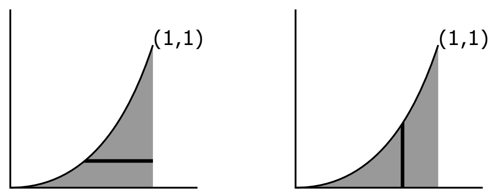
The previous paragraph is a complete solution. However, you can also use Lagrange multipliers to solve this problem. If you do that then you will find no solutions of the Lagrange equations $\vec{\nabla} f = \lambda \vec{\nabla} g$ that also satisfy the constraint, but you will find that the point $(0, 0)$ is a solution to $\vec{\nabla} g(x, y) = \vec{0}$ that also satisfies the constraint.

Thus for this problem the exceptional case leads you to the solution.

(3) Consider the double integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sin(\pi x^3)\, dx\, dy$

(a) (7%) Draw the domain of integration.

$\sqrt{y} \le x \le 1$



$0 \le y \le x^2$

(b) (18%) Compute the integral. The domain of integration is the region between the parabola $\sqrt{y} = x$ and the x-axis, with $0 \le x \le 1$.

There is no easy antiderivative for $\int \sin(\pi x^3)dx$ so we switch the order of integration and see what happens. We can describe the region as all points (x, y) with

$$0 \le y \le 1 \text{ and } \sqrt{y} \le x \le 1,$$

but also as all points (x, y) with

$$0 \le x \le 1 \text{ and } 0 \le y \le x^2.$$

This leads to the following integral

$$\begin{aligned}\int_{x=0}^{1} \int_{y=0}^{x^2} \sin(\pi x^3)\, dy\, dx &= \int_{x=0}^{1} \left[y \sin(\pi x^3)\right]_{y=0}^{x^2} dx \\ &= \int_{x=0}^{1} x^2 \sin(\pi x^3)\, dx \\ &= \left[-\frac{1}{3\pi} \cos(\pi x^3)\right]_{x=0}^{1} \\ &= \frac{2}{3\pi}.\end{aligned}$$

(4) (25%) Compute the triple integral

$$I = \iiint_{\mathcal{R}} \frac{dV}{x^2 + y^2 + z^2}$$

where $\mathcal{R}$ is the region *above the xy-plane,* and *inside the sphere* with radius R, and with the origin as center.

In spherical coordinates the region $\mathcal{R}$ is given by

$$0 \le \rho \le R, \qquad 0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \frac{\pi}{2}.$$

Using $x^2 + y^2 + z^2 = \rho^2$ and $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ we get

$$\begin{aligned}I &= \iiint_{\mathcal{R}} \frac{dV}{x^2 + y^2 + z^2} \\ &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{R} \frac{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}{\rho^2} \\ &= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{R} \sin\phi\, d\rho\, d\phi\, d\theta \\ &= 2R\pi.\end{aligned}$$