(1) (25%) Let f be the function $z = f(x, y) = x^2y - x - y^2$.

(a) Find the equation for the tangent plane to the graph of f at the point (x_0, y_0, z_0) with $x_0 = 1$, $y_0 = 3$.

Solution. The equation is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
Here we have $x_0 = 1$, $y_0 = 3$, $z_0 = f(x_0, y_0) = 1^2 \cdot 3 - 1 - 3^2 = -7$.
$f_x(x, y) = 2xy - 1 \implies f_x(x_0, y_0) = 2 \cdot 1 \cdot 3 - 1 = 5$
$f_y(x, y) = x^2 - 2y \implies f_x(x_0, y_0) = 1 - 6 = -5$
so the equation is

$$z + 7 = 5(x - 1) - 5(y - 3).$$

(b) Find the equation for the tangent to the level set of f at the point (x_0, y_0) with $x_0 = 1$, $y_0 = 3$.

Solution. The equation is

$$0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

which you get from $\vec{\nabla} f(x_0, y_0) \cdot (\vec{\boldsymbol{x}} - \vec{\boldsymbol{a}}) = 0$.
Using the values for $f_x(x_0, y_0)$, etc that were found in (a), you get

$$0 = 5(x - 1) - 5(y - 3)$$

which you can simplify to $y = x + 2$.

(2) (25%) For which values of the constants A, B, and C does there exist a function $z = f(x, y)$ with

$$\frac{\partial f}{\partial x} = Ax^2 + 2xy + y, \text{ and } \frac{\partial f}{\partial y} = Bx^2 + y + Cx \quad ?$$

(a) For those A, B, C for which no such function can be found, explain why this is so.

Solution. Clairaut says that the mixed partials must be equal, i.e.

$$f_{xy} = f_{yx}, \text{ i.e. } 2x + 1 = 2Bx + C.$$

This is only the case if $B = 1$ and $C = 1$. There is no restriction on A.

If either $B \neq 1$ or $C \neq 1$ (or both) then $f_{xy} \neq f_{yx}$, contradicting Clairaut's theorem, so that the function f cannot exist in those cases.

(b) Find the function $f(x, y)$ for those A, B, C for which it exists.

Solution. Assume $B = C = 1$. Then

$$f_x = Ax^2 + 2xy + y \implies f(x, y) = \frac{A}{3}x^3 + x^2y + xy + G(y)$$

for some function $G(y)$. Hence

$$f_y = x^2 + x + G'(y)$$

On the other hand we are given that

$$f_y = x^2 + y + x$$

so $G'(y) = y$, and $G(y) = \frac{1}{2}y^2 + K$ for some constant K. The function f is therefore

$$f(x, y) = \frac{A}{3}x^3 + x^2y + xy + \frac{1}{2}y^2 + K.$$

(3) (20%) Since 12 and 9 both are roughly equal to 10, one could guess that $\sqrt[5]{(12)^2 \cdot 9^3} \approx \sqrt[5]{10^5} = 10$. Find a more accurate approximation by applying the linear approximation formula to a suitably chosen function of two variables.

Indicate which function you are using and how you are applying the linear approximation formula.

Solution. Use the function $f(x, y) = \sqrt[5]{x^2 y^3} = x^{2/5} y^{3/5}$.

We want to approximate $f(12, 9) = f(10 + \Delta x, 10 + \Delta y)$ with $\Delta x = 2$, $\Delta y = -1$. The linear approximation formula says

$$f(12, 9) \approx f(10, 10) + f_x(10, 10)\Delta x + f_y(10, 10)\Delta y.$$

The partial derivatives are

$$\begin{aligned} f_x &= \frac{2}{5} x^{-3/5} y^{3/5} = \frac{2}{5} 10^{-3/5} 10^{3/5} = \frac{2}{5} \\ f_y &= \frac{3}{5} x^{2/5} y^{-2/5} = \frac{3}{5} 10^{2/5} 10^{-2/5} = \frac{3}{5} \end{aligned}$$

Therefore

$$f(12, 9) \approx f(10, 10) + \frac{2}{5} \cdot 2 + \frac{3}{5} \cdot (-1) = 10\tfrac{1}{5}$$

(4) (30%)

$$g(u, v) = f(x, y) \text{ where } x = u + v^2, \text{ and } y = uv.$$

Compute $g_u(u, v)$ and $g_{uv}(u, v)$ in terms of f and its derivatives,

Solution.

$$\begin{aligned} g_u &= \frac{\partial f(u + v^2, uv)}{\partial u} \\ &= f_x(u + v^2, uv) + f_y(u + v^2, uv)v \\ &= f_x + f_y v \text{ for short.} \end{aligned}$$

Hence, keeping in mind that f_x and f_y are evaluated at $(x, y) = (u + v^2, uv)$,

$$\begin{aligned} g_{uv} &= \frac{\partial f_x}{\partial v} + \frac{\partial f_y}{\partial v} v + f_y \frac{\partial v}{\partial v} \\ &= f_{xx} 2v + f_{xy} u + f_{yx} 2v \cdot v + f_{yy} u \cdot v + f_y \\ &= 2v f_{xx} + (u + 2v^2) f_{xy} + uv f_{yy} + f_y. \end{aligned}$$