math 234 — the first midterm — lecture 1

september 30, 2015

1. (25%) Let two points $A(2,2,1)$ and $B(2,0,3)$ be given

(a) Find a parametric representation for the line through A and B.

Solution. We need a point on the line (we choose A, although B would be just as good).

We need a vector in the direction of the line. Here we can choose either $\overrightarrow{AB}$ or $\overrightarrow{BA}$. Let's choose the first.

We get

$$\vec{\boldsymbol{a}} = \overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} = \begin{pmatrix} 2-2 \\ 0-2 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

A parametric equation for the line is

$$\vec{\boldsymbol{x}}(t) = \vec{\boldsymbol{a}} + t\overrightarrow{AB} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2-2t \\ 1+2t \end{pmatrix}.$$

(b) Find an equation for the plane through A, B, and the origin.

Solution. We have three points on the plane A, B, and the origin $O(0,0,0)$ From these we compute a normal

$$\vec{\boldsymbol{n}} = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -4 \end{pmatrix}.$$

The equation for the plane is then $\vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{x}} = \vec{\boldsymbol{n}} \cdot \vec{\boldsymbol{p}}$, where $\vec{\boldsymbol{p}}$ is the position vector of any point we know to be on the plane: we can choose $\vec{\boldsymbol{p}}$ to be either $\overrightarrow{OA}$, $\overrightarrow{OB}$, or $\vec{\boldsymbol{0}}$ (the position vector of the origin). We choose $\vec{\boldsymbol{p}} = \vec{\boldsymbol{0}}$ because that is the easiest, but the other choices should give the same answer. Thus we find the equation for the plane to be

$$-x_1 - 4x_2 - 4x_3 = 0.$$

If you don't like all the minus signs, then you can multiply both sides with -1, which gives you $x_1 + 4x_2 + 4x_2 = 0$.

2. (30%)The vector function $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} t \\ 1-t^2 \\ 1+t^2 \end{pmatrix}$ describes the motion of a point $X(t)$ in space.

(a) Compute the velocity, speed, and acceleration of the point at any time t.

Solution. Velocity : $\vec{\boldsymbol{x}}'(t) = \begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix}$.

Speed is length of velocity: $\|\vec{\boldsymbol{x}}'(t)\| = \sqrt{1 + (-2t)^2 + (2t)^2} = \sqrt{1+8t^2}$.

Acceleration (always a vector) : $\vec{\boldsymbol{x}}''(t) = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$.

(b) Where does the tangent line to the curve at the point $X(t)$ with $t = 1$ intersect the xy plane?

Solution. A parametrization for the tangent line at $X(1)$ is

$$\vec{\boldsymbol{y}}(s) = \vec{\boldsymbol{x}}(1) + s\vec{\boldsymbol{x}}'(1) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+s \\ -2s \\ 2+2s \end{pmatrix}$$

This vector is on the xy-plane when its z-component is zero, i.e. when $2 + 2s = 0$. That happens when $s = -1$, so the intersection with the xy-plane is at

$$\vec{\boldsymbol{y}}(-1) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

(c) Find the unit tangent vector to the curve at $X(t)$ when $t = 1$.

Solution. A tangent vector is $\vec{\boldsymbol{x}}'(1) = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. The length of that vector is

$$\|\vec{\boldsymbol{x}}'(1)\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3.$$

To get a unit vector with the same direction, divide $\vec{\boldsymbol{x}}'(1)$ by its length

$$\vec{\boldsymbol{T}} = \frac{\vec{\boldsymbol{x}}'(t)}{\|\vec{\boldsymbol{x}}'(t)\|} = \begin{pmatrix} 1/3 \\ -2/3 \\ 2/3 \end{pmatrix}.$$

(d) Find an integral for the length of the part of the curve for which $0 \leq t \leq 1$ (no need to compute the integral).

Solution. The length of the path from $X(0)$ to $X(1)$ traced out by the vector function $\vec{\boldsymbol{x}}(t)$ is the "distance traveled," which is

$$\text{distance traveled} = \int_{t=0}^{1} \text{Speed}\, dt = \int_0^1 \sqrt{1 + 8t^2}\, dt.$$

3. (30%) For each of the following functions

- decide if they are positive definite, negative definite, indefinite, or semidefinite quadratic forms.
- *draw the zero set* of the function, and indicate *the region where the function is positive,* and *the region where the function is negative.*

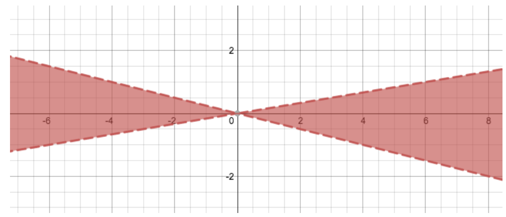
(a) $f(x, y) = x^2 - 2xy - 24y^2$

(b) $g(x, y) = xy - 3y^2$

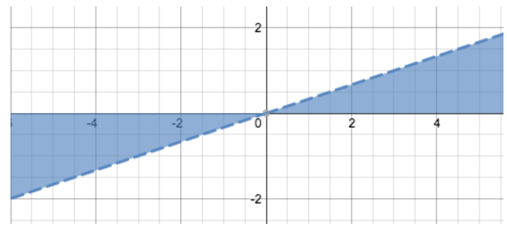
(c) $h(x, y) = -x^2 + 4xy - 4y^2$

Solution.

$$\begin{aligned} f(x,y) &= x^2 - 2xy - 24y^2 \\ &= (x-y)^2 - y^2 - 24y^2 \\ &= (x-y)^2 - 25y^2 \\ &= (x-y-5y)(x-y+5y) \\ &= (x-6y)(x+4y) \end{aligned}$$

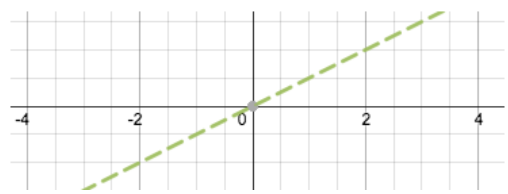


So, $f(x,y)$ is indefinite.



$g(x,y) = xy - 3y^2 = (x-3y)y$
Conclusion: $g(x,y)$ is also indefinite.

$$\begin{aligned} h(x,y) &= -x^2 + 4xy - 4y^2 \\ &= -\left(x^2 - 4xy + 4y^2\right) \\ &= -(x-2y)^2. \end{aligned}$$



This shows that $h(x,y)$ is semidefinite. Because of the minus sign you can be more precise and say that $h(x,y)$ is *negative semidefinite.*
$h(x,y)$ is everywhere negative, except on its zeroset, which is the line $x = 2y$.

4. (15%)

Suppose a point $X(t)$ is moving through space, and suppose its position vector at time t is $\vec{\boldsymbol{x}}(t)$.

At first the point is moving slowly. Then it speeds up, and later it slows down again. We are given that the point reaches its highest speed at time $t = t_0$.

Explain why at the moment $t = t_0$ the velocity and acceleration of the point are perpendicular.

Suggestion: first explain why the function $\|\vec{\boldsymbol{x}}'(t)\|^2$ has a maximum at time $t = t_0$.

Solution. The speed of the point is $\|\vec{\boldsymbol{x}}'(t)\|$. It is a function of time t, and it reaches its maximum value at $t = t_0$.

The square of the speed therefore also reaches its maximal value at time $t = t_0$.

When a function reaches its maximum its derivative is zero, so at the moment $t = t_0$ we know that

$$\frac{d\|\vec{\boldsymbol{x}}'(t)\|^2}{dt} = 0.$$

We can expand the derivative by using $\|\vec{\boldsymbol{a}}\|^2 = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}}$. We find that at any time t we have

$$\frac{d\|\vec{\boldsymbol{x}}'(t)\|^2}{dt} = \frac{d\,\vec{\boldsymbol{x}}'(t) \cdot \vec{\boldsymbol{x}}'(t)}{dt} = \vec{\boldsymbol{x}}'(t) \cdot \vec{\boldsymbol{x}}''(t) + \vec{\boldsymbol{x}}''(t) \cdot \vec{\boldsymbol{x}}'(t) = 2\vec{\boldsymbol{x}}'(t) \cdot \vec{\boldsymbol{x}}''(t).$$

When $t = t_0$ we have $\frac{d\|\vec{\boldsymbol{x}}'\|^2}{dt} = 0$, so at that time we also have

$$\vec{\boldsymbol{x}}'(t_0) \cdot \vec{\boldsymbol{x}}''(t_0) = 0.$$

This implies that velocity and acceleration are perpendicular at time t_0.