Here are some solutions to the final exam from two years ago: `http://www.math.wisc.edu/~angenent/234.2015f/final/exam2013-with-some-solutions.pdf`

(1) (a)

$$\begin{aligned} f_x(x,y) &= e^{x-2y} + xe^{x-2y} \\ f_y(x,y) &= -2xe^{x-2y} \end{aligned}$$

So, we have at the point $x = 2$, $y = 1$

$$\begin{aligned} f(2,1) &= 2 \\ f_x(2,1) &= 3 \\ f_y(2,1) &= -4 \end{aligned}$$

Thus the linear approximation of $f(x,y)$ near $x = 2, y = 1$ is

$$f(x,y) \approx 2 + 3(x-2) - 4(y-1).$$

(b)

$$f(1.99, 1.02) \approx 1.89$$

(3) (a)

$$\begin{aligned} f_x(x,y) &= 2x \\ f_y(x,y) &= -6y + 3y^2 \end{aligned}$$

So there are two critical points: $(0,0)$ and $(0,2)$.

(b)

$$\begin{aligned} f_{xx}(x,y) &= 2 \\ f_{xy}(x,y) &= 0 \\ f_{yy}(x,y) &= -6 + 6y \end{aligned}$$

At the critical point $(0,0)$, the quadratic part in Taylor's expansion is

$$\begin{aligned} Q(\Delta x, \Delta y) &= \frac{1}{2}(2\Delta x^2 - 6\Delta y^2) \\ &= \Delta x^2 - 3\Delta y^2 \\ &= (\Delta x - \sqrt{3}\Delta y)(\Delta x + \sqrt{3}\Delta y) \end{aligned}$$

So $(0,0)$ is a saddle point. Since, $\Delta x = x - 0$ and $\Delta y = y - 0$, the equations of the tangent lines to the level set are

$$\begin{aligned} x - \sqrt{3}y &= 0 \\ x + \sqrt{3}y &= 0 \end{aligned}$$

At the critical point $(0,2)$,

$$\begin{aligned} Q(\Delta x, \Delta y) &= \frac{1}{2}(2\Delta x^2 + 6\Delta y^2) \\ &= \Delta x^2 + 3\Delta y^2 \end{aligned}$$

So $(0,2)$ is a local minimum.

(4) $f(x,y) = xy^2$, $g(x,y) = x^2 + y^2 = 3$. At the points satisfying $\vec{\nabla} g = \vec{0}$, we cannot use Lagrange multiplier. $g_x = 2x$ and $g_y = 2y$, so $(0,0)$ is the only point making $\vec{\nabla} g$ zero. But this point $(0,0)$ does not satisfy $g(x,y) = 3$, which means it's not on the constraint set we are considering.

Now for all points on the constraint set, we could apply Lagrange multiplier,

$$\begin{cases} y^2 = \lambda 2x \\ 2xy = \lambda 2y \\ x^2 + y^2 = 3 \end{cases}$$

There are six solutions $(x,y,\lambda) = (\sqrt{3},0,0), (-\sqrt{3},0,0), (1,\sqrt{2},1), (1,-\sqrt{2},1),$ $(-1,\sqrt{2},-1)$ and $(-1,-\sqrt{2},-1)$. So the maxima are at points $(1,\sqrt{2})$, $(1,-\sqrt{2})$, and the minima are at the point $(-1,\sqrt{2})$, $(-1,-\sqrt{2})$.

(7) (a) the region $\mathcal{R}$ in cylindrical coordinates is presented by

$$\begin{aligned} 0 \leq\ & r \leq 2 \\ \frac{\pi}{4} \leq\ & \theta \leq \frac{\pi}{2} \\ 0 \leq\ & z \leq r^2 \end{aligned}$$

(b)

$$\begin{aligned}\iiint_{\mathcal{R}} 1\ dV &= \int_0^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{r^2} r\ dz\ d\theta\ dr \\ &= \pi \\ \iiint_{\mathcal{R}} z\ dV &= \int_0^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{r^2} zr\ dz\ d\theta\ dr \\ &= \frac{4}{3}\pi\end{aligned}$$

So the average of z in $\mathcal{R}$ is

$$\frac{\iiint_{\mathcal{R}} z\ dV}{\iiint_{\mathcal{R}} 1\ dV} = \frac{4}{3}.$$

(8)

$$\int_0^3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \rho^3 \sin^2\phi \cos\theta\ d\phi\ d\theta\ d\rho.$$

(9) See problem 1 on Dec 10's handout.

Here are solutions to 8,9,10 on `http://www.math.wisc.edu/~angenent/234.2015f/final/lineintegralproblems.html`

(8) $\vec{T}\,ds = d\vec{x} = \begin{pmatrix} dx \\ dy \end{pmatrix}$, and because $\vec{N}$ is the unit normal obtained by rotating $\vec{T}$ clockwise by

$$\vec{N}\,ds = \begin{pmatrix} dy \\ -dx \end{pmatrix}.$$

Thus, we could see that

$$\vec{F} = \vec{H} = \begin{pmatrix} \sin(x) \\ e^{xy} \end{pmatrix}, \text{ and } \vec{G} = \begin{pmatrix} e^{xy} \\ -\sin(x) \end{pmatrix}.$$

(9) Apply Green Theorem.

$$\begin{aligned} I &= \int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds \\ &= -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA \\ &= -\iint_R 0\,dA = 0 \end{aligned}$$

We have a negative sign above is because the curve $\mathcal{C}$ is in the clockwise orientation. And for J, since $\vec{N}$ is the outward unit normal,

$$\begin{aligned} J &= \int_{\mathcal{C}} \vec{F}\cdot\vec{N}\,ds \\ &= \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA \\ &= \iint_R (x^2+y^2)\,dA \\ &> 0 \end{aligned}$$

The last step is because the integrand $x^2+y^2 \geq 0$.

(10) Because $\vec{T}\cdot\vec{T} = \|\vec{T}\|^2 = 1$, $\vec{N}\cdot\vec{N} = \|\vec{N}\|^2 = 1$ and $\vec{T}\cdot\vec{N} = 0$, we have $I_1 = I_2 = \int_{\mathcal{C}} 1\,ds$ represent the length of $\mathcal{C}$ and $I_3 = 0$