math 234 final exam, fall 2013
with some solutions

(1) Let $f(x,y) = xe^{x-2y}$.
 (a) Find the linear approximation of $f(x,y)$ near $x = 2, y = 1$.
 (b) Use the linear approximation to approximate $f(1.99, 1.02)$.

(2) Suppose we are given a function $f(x,y)$. Define $g(u,v) = f(u\ln v, u+v)$. In the following questions your answer may contain the variables u and v as well the partial derivatives of f with respect to x and/or y.
 (a) Compute $\dfrac{\partial g}{\partial u}$.
 (b) Compute $\dfrac{\partial^2 g}{\partial u \partial v}$.

For part (a):

$$\begin{aligned}
\frac{\partial g}{\partial u} &= \frac{\partial f(\overbrace{u\ln v}^{x}, \overbrace{u+v}^{y})}{\partial u} \\
&= f_x(u\ln v, u+v)\frac{\partial\, u\ln v}{\partial u} + f_y(u\ln v, u+v)\frac{\partial\, u+v}{\partial u} \\
&= f_x(u\ln v, u+v)\cdot \ln v + f_y(u\ln v, u+v)\cdot 1 \\
&= f_x(u\ln v, u+v)\ln v + f_y(u\ln v, u+v)
\end{aligned}$$

For part (b):

$$\frac{\partial^2 g}{\partial u\partial v} = \frac{\partial \frac{\partial g}{\partial u}}{\partial v}$$

so we are asked to differentiate $g_u(u,v)$, which we found in part (a), with respect to v. Our formula for g_u has two terms, the first of which is a product of $f_x(\cdots)\ln v$. We differentiate each term and apply the product rule to the first term:

$$\begin{aligned}
\frac{\partial^2 g}{\partial u\partial v} &= \frac{\partial \frac{\partial g}{\partial u}}{\partial v} \\
&= \frac{\partial\{f_x(u\ln v, u+v)\ln v + f_y(u\ln v, u+v)\}}{\partial v} \\
&= \frac{\partial\, f_x(u\ln v, u+v)\ln v}{\partial v} + \frac{\partial\, f_y(u\ln v, u+v)}{\partial v} \\
&= \frac{\partial\, f_x(u\ln v, u+v)}{\partial v}\ln v + f_x(u\ln v, u+v)\frac{\partial \ln v}{\partial v} + \frac{\partial\, f_y(u\ln v, u+v)}{\partial v} \\
&= \frac{\partial\, f_x(u\ln v, u+v)}{\partial v}\ln v + f_x(u\ln v, u+v)\frac{1}{v} + \frac{\partial\, f_y(u\ln v, u+v)}{\partial v}
\end{aligned}$$

The two remaining derivatives are done in exactly the same way as in part (a). We get

$$\frac{\partial\, f_x(u\ln v, u+v)}{\partial v} = f_{xx}(u\ln v, u+v)\frac{u}{v} + f_{xy}(u\ln v, u+v)1$$

and

$$\frac{\partial\, f_y(u\ln v, u+v)}{\partial v} = f_{yx}(u\ln v, u+v)\frac{u}{v} + f_{yy}(u\ln v, u+v)1$$

Putting these two derivatives in our previous expression for $\partial^2 g/\partial u\partial v$, we get

$$\begin{aligned}\frac{\partial^2 g}{\partial u\partial v} =& \Big\{ f_{xx}(u\ln v, u+v)\frac{u}{v} + f_{xy}(u\ln v, u+v)\Big\}\ln v + f_x(u\ln v, u+v)\frac{1}{v} \\ &+ \Big\{ f_{yx}(u\ln v, u+v)\frac{u}{v} + f_{yy}(u\ln v, u+v)\Big\}.\end{aligned}$$

This is the result. We can clean it up a little by using $f_{xy} = f_{yx}$ and combining like terms:

$$\frac{\partial^2 g}{\partial u\partial v} = \frac{u\ln v}{v} f_{xx} + \big\{\ln v + \frac{u}{v}\big\} f_{xy} + f_{yy} + \frac{1}{v} f_x\,,$$

where we have left out the "$(u\ln v, u+v)$" after f_{xx}, etc.

(3) (a) Find all critical points of the function $f(x, y) = x^2 - 3y^2 + y^3$;
(b) Use the second derivative test to tell which of the critical points are local maxima, minima, or saddle points. For any saddle points find the tangent lines to the level set.

(4) Use the method of Lagrange multipliers to find the largest and smallest values that xy^2 can have if $x^2 + y^2 = 3$.

(5) Let V be the volume under the graph of

$$z = \frac{1}{(x+y)^2}$$

above the rectangle in which $3 \le x \le 6$ and $0 \le y \le 2$.
(a) Which integral do you have to compute to find V?
(b) Compute V.

In this case the domain is a rectangle, so the integration bounds are constants. We have to compute

$$\begin{aligned}V &= \int_{y=0}^{2}\int_{x=3}^{6} \frac{1}{(x+y)^2}\,dx\,dy \\ &= \int_{y=0}^{2} \Big[\frac{-1}{x+y}\Big]_{x=3}^{6} dy \\ &= \int_{y=0}^{2} \Big(\frac{-1}{y+6} - \frac{-1}{y+3}\Big) dy \\ &= \Big[-\ln(y+6) + \ln(y+3)\Big]_{y=0}^{2} \\ &= -\ln 8 + \ln 5 + \ln 6 - \ln 3 = \ln\frac{5\cdot 6}{8\cdot 3} = \ln\frac{5}{4}.\end{aligned}$$

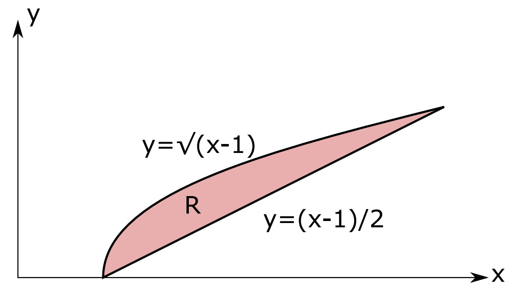
(6) Consider the integral

$$I = \iint_{\mathcal{R}} f(x, y)\,dA = \int_1^5 \int_{(x-1)/2}^{\sqrt{x-1}} f(x, y)\,dy\,dx$$

(a) Draw the domain $\mathcal{R}$

(b) Find the integration bounds that appear when you rewrite the integral in the form

$$I = \int_{\cdots}^{\cdots} \int_{\cdots}^{\cdots} f(x,y)\, dx\, dy$$



The domain is the region contained between the parabola $y = \sqrt{x-1}$ and the line $y = (x-1)/2$. One can also describe the parabola and line with the equations $x = 1 + y^2$ and $x = 2y + 1$. The integral is

$$I = \int_{y=0}^{2} \int_{x=y^2+1}^{2y+1} f(x,y)\, dx\, dy$$

(7) Let $\mathcal{R}$ be the three dimensional region given by $x \geq 0$, $y \geq x$, and $x^2 + y^2 \leq 4$, and $0 \leq z \leq x^2 + y^2$.

(a) Describe $\mathcal{R}$ in cylindrical coordinates (r, θ, z).

(b) Compute the average of z in $\mathcal{R}$.

(8) Let $\mathcal{R}$ be the three dimensional region inside the sphere $x^2 + y^2 + z^2 \leq 9$ that is given by $x \geq 0$, $y \geq 0$, $z \geq 0$, and $y \geq x$.

(a) Make a drawing that describes the spherical coordinates (ρ, ϕ, θ) of a point and describe the region $\mathcal{R}$ in spherical coordinates.

(b) Write the integral

$$I = \iiint_{\mathcal{R}} x\, dV$$

in terms of spherical coordinates. Do not compute the integral.

(9) Let A, B, and C be the three points $A(0,0)$, $B(1,1)$, $C(2,0)$.

Let $\vec{\boldsymbol{F}}$ be the vectorfield $\vec{\boldsymbol{F}}(x,y) = \binom{-1}{1}$.

(a) Compute $\int_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{T}}\, ds$ where $\mathcal{C}$ is the curve that consists of the line segment AB followed by the line segment BC (i.e. $\mathcal{C}$ is the top of the triangle ABC, oriented from left to right.)

(b) Compute $\int_{\mathcal{C}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}}\, ds$, where $\vec{\boldsymbol{N}}$ is the unit normal to $\mathcal{C}$ that points upward.

(c) Use Green's theorem to compute the line integral

$$I = \int_{\mathcal{T}} \vec{\boldsymbol{G}} \cdot \vec{\boldsymbol{T}}\, ds$$

where $\mathcal{T}$ is the triangle ABC with counter-clockwise orientation, and $\vec{\boldsymbol{G}}$ is the vector field

$$\vec{\boldsymbol{G}}(x,y) = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$