

## Study guide for the second midterm

### Topics covered

**The chain rule and the linear approximation formula.** You should understand the linear approximation formula, how it implies the chain rule, and you should know how to use the chain rule to compute derivatives of first and second order of a function.

**Higher derivatives and Clairaut's Theorem.** Know that mixed partials are equal, i.e.

$$f_{xy} = f_{yx}$$

(provided they are continuous). Know when two given functions  $P(x, y)$  and  $Q(x, y)$  can be the partial derivatives of another function  $f(x, y)$ , and know how to find that function  $f$  when it exists.

**Optimization.** Know the basic theorems

- If  $\vec{\nabla} f \neq \vec{0}$  at some point  $(a, b)$ , then the level set of  $f$  through the point  $(a, b)$  is a curve (at least near the point  $(a, b)$ ), and the gradient  $\vec{\nabla} f(a, b)$  is perpendicular to the level set at  $(a, b)$ .
- If a function  $f$  is continuous on a region  $R$ , and if that region is bounded and if the region includes all its boundary points, then the function attains both a maximum and a minimum in the region  $R$ .
- If a function is differentiable in a region  $R$ , and if it attains a local minimum or maximum at an interior point  $(a, b)$ , then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ .

Using these theorems you should be able to detect saddle points and predict maxima and minima for functions whose zero set you know.

**Taylor's formula and the Second Derivative Test.** This particular topic is for functions of two variables only:

- Know the second order Taylor expansion for a function of two variables.
- Know how to use the 2nd order Taylor formula to tell if a critical point is a local minimum, local maximum, saddle point, or none of these.

**Optimization problems with constraints.** You should be able to use the method of Lagrange Multipliers to find local minima or maxima of a function of two or three variables.

### Old 234 midterm problems with some solutions

1. For each of the following functions,
  - Draw the zero set; how many critical points can you detect by knowing the zeroset of the function?
  - Find all critical points of the function  $z = f(x, y)$ .
  - At each critical point compute the Taylor expansion of order two of the function.
  - Classify the critical points you found above as local minimum, local maximum, saddle, or other type. For saddle points find the two tangents to the level curves through the saddle point.

(a)  $f(x, y) = (4x - x^2 - y^2)y$ .

(b)  $f(x, y) = (4x - x^2 - y)y$ .

2. Find all the critical points of the function  $f(x, y) = x^2y(1 - x - y)$ .
3. Find the critical points of the function  $f(x, y) = \sqrt{xy}(1 - x - y)$  for which one has  $x > 0, y > 0$ .
4. Find the critical point(s) of the function  $f(x, y) = xye^{-x^2-y^2}$  for which one has  $x > 0, y > 0$ .
5. Find the critical points of the function  $f(x, y) = xye^{-ax-by}$ , in which  $a$  and  $b$  are positive constants.
6. Find the critical points of the function  $f(x, y) = x^ay^be^{-x-y}$ , in which  $a$  and  $b$  are positive constants.
7. (a) For which values of the constants  $A$  and  $B$  is it possible to find a function  $f(x, y)$  for which both

$$\frac{\partial f}{\partial x} = Axy + B \cos x \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + \sin y$$

hold? For those values of  $A$  and  $B$  for which you claim that no such function can be found, explain why this is not possible.

For values of  $A$  and  $B$  for which the function  $f$  can be found explain how you find it, and compute the function  $f$ .

- (b) Same question for

$$\frac{\partial f}{\partial x} = Axy + y^B \cos x \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + \sin x$$

- (c) Same for

$$f_x(x, y) = x^A y, \quad f_y(x, y) = xy^B.$$

8. Let  $f(x, y)$  be a function of two variables. If  $F(r, \theta) = f(r \cos \theta, r \sin \theta)$  then compute

$$F_\theta \text{ and } F_{\theta\theta}$$

in terms of  $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ .

9. (a) Find the critical points of the function  $f(x, y) = -y^2 + xy - x^2y$ .  
 (b) What is the largest value  $f(x, y) = -y^2 + xy - x^2y$  can have if  $0 \leq x \leq 1$ ? (Explain your answer – hint: at which points  $(x, y)$  is  $f(x, y) = 0$ , and where is  $f(x, y) > 0$ ?)
10. Use the method of Lagrange multipliers to find the points  $(x, y)$  satisfying  $x^2 + 4y^2 = 18$  at which  $f(x, y) = x + 2y$  is maximal or minimal.
11. Use the method of Lagrange multipliers to find the points  $(x, y)$  satisfying  $4x^2 + y^2 = 18$  at which  $f(x, y) = x + y/2$  is maximal or minimal.

12. Let  $\mathcal{C}$  be the curve with equation  $x^2 + y^2 = 4$ .
- (a) Draw  $\mathcal{C}$ .
  - (b) Use the method of Lagrange multipliers to find the maximal and minimal values of  $f(x, y) = y - x^2$  on  $\mathcal{C}$ .
13. The ACME*boxes* company makes rectangular boxes in which the bottom and top are made of material that costs \$3 per square inch, and for which the four vertical sides cost \$2 per square inch. This problem asks for the shape of the cheapest box that ACME*boxes* can make with volume  $V$ .
- (a) Formulate this problem as an minimization problem with constraints. Which (and how many) variables do you choose? What is the function that you minimize, and which function describes the constraint?
  - (b) Use the method of Lagrange multipliers to solve the minimization problem.
14. The ACME*cylinders* company makes cylindrical boxes in which the bottom and top are made of material that costs \$3 per square inch, and for which the vertical cylindrical side cost \$2 per square inch. This problem asks for the shape of the cheapest box that ACME*cylinders* can make with volume  $V$ .
- (a) Formulate this problem as an minimization problem with constraints. Which (and how many) variables do you choose? What is the function that you minimize, and which function describes the constraint?
  - (b) Use the method of Lagrange multipliers to solve the minimization problem.
15. Use Lagrange multipliers to find the smallest value of

$$f(x, y, z) = 2xy + yz + xz$$

on the surface given by

$$xyz = 1$$