12. Let $\mathcal{C}$ be the curve with equation $x^2 + y^2 = 4$.

(a) Draw $\mathcal{C}$.

(b) Use the method of Lagrange multipliers to find the maximal and minimal values of $f(x, y) = y - x^2$ on $\mathcal{C}$.

Solution: The constraint is $g(x, y) = 4$, where $g(x, y) = x^2 + y^2$.

Minima and Maxima will be found at solutions of the Lagrange multiplier equation, or at points with

$$g(x, y) = 2, \quad g_x(x, y) = 0, \quad g_y(x, y) = 0.$$

Since $g_x = 2x$ and $g_y = 2y$ the only point with $g_x = g_y = 0$ is the origin $x = y = 0$. But this point does not satisfy the constraint equation. Therefore any maximum or minimum must be at a solution of the Lagrange multiplier equations.

The Lagrange multiplier equations are

$$\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y), \text{ and } g(x, y) = 4,$$

Since

$$f_x = -2x, \quad f_y = 1, \qquad g_x = 2x, \quad g_y = 2y,$$

we have to solve

$$-2x = \lambda\, 2x, \quad 1 = \lambda\, 2y, \quad x^2 + y^2 = 4.$$

The first equation implies either $x = 0$ or $\lambda = -1$.

Case 1: $x = 0$. In this case the constraint implies $y = \pm 2$, and the second equation ($f_y = \lambda g_y$) implies $\lambda = \pm\frac{1}{4}$.

Case 2: $\lambda = -1$. Now the second equation implies $1 = -2y$, so that $y = -\frac{1}{2}$. The constraint implies $x^2 = 4 - y^2 = 3\frac{3}{4} = \frac{15}{4}$, and thus $x = \pm\frac{1}{2}\sqrt{15}$.

So we find four critical points:

$$\begin{array}{ll}
(0, 2) & f(0, 2) = 2 \\
(0, -2) & f(0, -2) = -2 \\
(\frac{1}{2}\sqrt{15}, -\frac{1}{2}) & f(\frac{1}{2}\sqrt{15}, -\frac{1}{2}) = -\frac{1}{2} - \frac{15}{4} = -\frac{17}{4} \\
(-\frac{1}{2}\sqrt{15}, -\frac{1}{2}) & f(-\frac{1}{2}\sqrt{15}, -\frac{1}{2}) = -\frac{17}{4}
\end{array}$$

The ones with the lowest function value are the minima, the one with the highest function value is the maximum:

Minima at $(\pm\frac{1}{2}\sqrt{15}, -\frac{1}{2})$

Maximum at $(0, 2)$.

13. The ACME*boxes* company makes rectangular boxes in which the bottom and top are made of material that costs \$3 per square inch, and for which the four vertical sides cost \$2 per square inch. This problem asks for the shape of the cheapest box that ACME*boxes* can make with volume V.

(a) Formulate this problem as an minimization problem with constraints. Which (and how many) variables do you choose? What is the function that you minimize, and which function describes the constraint?

(b) Use the method of Lagrange multipliers to solve the minimization problem.

Solution: If the bottom of the box has sides x and y, and if the height is z, then the total cost to make the box is

$$f(x, y, z) = 3xy + 4xz + 4yz + 3xy = 6xy + 4xz + 4yz.$$

This function is the one we want to minimize.

The constraint is given by

$$g(x, y, z) = xyz = V.$$

First look for points where

$$g = 0, \qquad \vec{\boldsymbol{\nabla}} g = \vec{\mathbf{0}}.$$

These equations are:

$$xyz = V, \qquad yz = 0,\ xz = 0,\ xy = 0.$$

If $xy = 0$ then $xyz = 0$ so such a point does not satisfy the constraint $xyz = V$.

So the equations $g = 0$, $\vec{\boldsymbol{\nabla}} g = \vec{\mathbf{0}}$ have no solutions. Any minima will be solutions of the Lagrange equations.

The derivatives of f and g are given by

$$f_x = 6y + 4z, \quad f_y = 6x + 4z, \quad f_z = 4x + 4y$$

$$g_x = yz, \quad g_y = xz, \quad g_z = xy.$$

Therefore the Lagrange equations $\vec{\boldsymbol{\nabla}} f = \lambda \vec{\boldsymbol{\nabla}} g$ are

$$\begin{aligned} 6y + 4z &= \lambda yz \\ 6x + 4z &= \lambda xz \\ 4x + 4y &= \lambda xy \\ xyz &= V \end{aligned}$$