Study guide for the first midterm

Topics covered

Vectors. Know
- Equation of lines and planes, distance to a line or plane if you know a point on the plane and a normal to the plane.
- how to compute the angle between two vectors, or the angles between two planes (which is the angle between their normals)
- the cross product; how to compute it, area of parallelogram or triangle, finding a vector perpendicular to two given vectors.

Parametric Curves. Be able to find:
- velocity, acceleration,
- tangent line to a curve (where does it intersect a given plane? Where does it have a specified direction?)
- length of a curve.

Quadratic forms. For a given quadratic form you should know how to...
- ...decide if it is positive definite, negative definite, semidefinite, or indefinite. (When you're studying, give an example of each kind of quadratic form.)
- ...complete the square and factor the form, if that is possible. If a form is indefinite, know how to find its zero set.

Partial derivatives. Show that you know how to compute f_x, f_y, $\vec{\nabla} f$ if you are given a formula for a function $f(x, y)$.

Linear approximation. Know that the change in a function $f(x, y)$ due to small changes Δx and Δy in x and y is approximately given by equation (59) or (60) (page 54.)

Be able to use this formula to approximate $f(x + \Delta x, y + \Delta y)$ if you know f, f_x, f_y at the point (x, y) (as in problem 1, Ch.4–§7)

The two variable Chain rule. Know the statement. Be able to answer questions like problems Ch.4-§12:1,2, and 3, from the text, and also problems 7 and 8 below.

Old 234 midterm problems with some solutions

Vector problems

1. Given: points $A(2, 1)$, $B(3, 2)$, $C(4, 4)$ and $D(5, 2)$. *Is $ABCD$ a parallelogram?* •

2. **(a)** Find the defining equation and a normal vector $\vec{n}$ for the line ℓ (in the plane) that is the graph of $y = 1 + \frac{1}{2}x$. •
(b) What is the distance from the origin to ℓ? •
(c) Answer the same two questions for the line m which is the graph of $y = 2 - 3x$. •
(d) What is the angle between ℓ and m? •

3. Given $A(2,0,0)$, $B(0,0,2)$ and $C(2,2,2)$. Let $\mathcal{P}$ be the plane through A, B and C.
(a) Find a normal vector for $\mathcal{P}$.
(b) Find a defining equation for $\mathcal{P}$.
(c) What is the distance from $D(0,2,0)$ to $\mathcal{P}$? What is the distance from the origin $O(0,0,0)$ to $\mathcal{P}$?
(d) Do D and O lie on the same side of $\mathcal{P}$?
(e) Find the area of the triangle ABC.
(f) Where does the plane $\mathcal{P}$ intersect the three coordinate axes?

4. **(a)** Does $D(2,1,3)$ lie on the plane $\mathcal{P}$ through the points $A(-1,0,0)$, $B(0,2,1)$ and $C(0,3,0)$?
(b) The point $E(1,1,\alpha)$ lies on $\mathcal{P}$. What is α?

Parametric Curves, Quadratic forms

See the homework problems in the text.

Partial derivatives, Linear approximation, Tangent planes

5. A function $z = f(x,y)$ satisfies $f(2,1) = 4$, $f_x(2,1) = -2$ and $f_y(2,1) = 2$. Approximate $f(2.01, 0.99)$ based on these data.

6. Find the equation for the tangent plane to the graph of $z = e^{x^2-y}$ at the point where $x = \sqrt{2}$ and $y = 2$.

7. Compute

$$\frac{df(\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)}{d\theta}$$

at $\theta = \pi/4$ if you know that the function f satisfies $f_x(1,1) = 4$ and $f_y(1,1) = 12$.

8. Let $z = f(x,y)$ be a function of two variables which satisfies

$$f(4,0) = 1, \qquad \frac{\partial f}{\partial x}(4,0) = 2, \qquad \frac{\partial f}{\partial y}(4,0) = -3.$$

(a) Compute $g_x(4,0)$ and $g_y(4,0)$ if $g(x,y) = \sin(\pi f(x,y))$.
(b) Compute $h'(0)$ if $h(t) = f(4\cos t, \sin t)$.

9. We are given a function $z = f(x,y)$. At the point $(x_0, y_0) = (-1,1)$ the function value is $z = 2$, and the gradient is $f_x(-1,1) = 2$ and $f_y(-1,1) = 3$. Approximate $f(-0.98, 1.01)$ based on these data.

10. Find the equation for the tangent plane to the graph of

$$z = \sin\left(-\pi x - 3\pi y\right)$$

at the point where $x = 1/4$ and $y = 1/6$.

11. Given a function $z = f(x,y)$ we consider the function

$$g(t) = f(t^2, 4t^4 + t^3 - t).$$

It is known that $f_x(1,4) = A$, and $f_y(1,4) = B$.
Compute $g'(-1)$ and $g'(1)$. (The constants A and B may appear in your answer.)

12. **(a)** Find the equation for the tangent plane to the graph of $z = f(x,y)$, where

$$f(x,y) = \ln\left(1 + x^2 + y^2\right)$$

at the point $(x_0, y_0) = (1,3)$.

(b) Where does the tangent plane intersect the z axis?

13. **(a)** Compute the partial derivatives of $f(x,y)=\dfrac{2x}{1+y^2}$. •

(b) Find an equation for the tangent plane to the graph of $z=f(x,y)$ at the point $(x_0,y_0)=(3,1)$ •

(c) Use linear approximation to compute $f(3.1,0.98)$ starting from $f(3,1)=3$. •

14. For a function $z=f(x,y)$ we are given the following values:

(x,y)	$f_x(x,y)$	$f_y(x,y)$
$(0,0)$	3	7
$(2,0)$	5	-2
$(0,3)$	2	-1

(a) Compute $h'(0)$ if h is the function given by $h(t)=f(2\cos t,\sin 3t)$. •

(b) Find a normal vector to the tangent line to the level set of the function f at the point $(0,3)$. Then find an equation for this tangent line. •

(c) If you find yourself at the point $(2,1,1)$, and if you want to ***decrease*** the function $g(x,y)$ as fast as possible, then in which direction should you move? (Give a vector in the direction in which you should move.) •

Some Solutions

(1) One always labels the vertices of a parallelogram counterclockwise (see §??).

$ABCD$ is a parallelogram if $\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC}$. $\overrightarrow{AB}=\begin{pmatrix}1\\1\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}2\\3\end{pmatrix}$, $\overrightarrow{AD}=\begin{pmatrix}3\\1\end{pmatrix}$. So $\overrightarrow{AB}+\overrightarrow{AD}\neq\overrightarrow{AC}$, and $ABCD$ is not a parallelogram.

(2a) ℓ has defining equation $-\frac{1}{2}x+y=1$ which is of the form $\vec{n}\cdot\vec{x}$ =constant if you choose $\vec{n}=\binom{-1/2}{1}$.

(2b) The distance to the point D with position vector $\vec{d}$ from the line ℓ is $\frac{\vec{n}\cdot(\vec{d}-\vec{a})}{\|\vec{n}\|}$ where $\vec{a}$ is the position vector of any point on the line. In our case $\vec{d}=\vec{0}$ and the point $A(0,1)$, $\vec{a}=\overrightarrow{OA}=\binom{0}{1}$, is on the line. So the distance to the origin from the line is $\dfrac{-\vec{n}\cdot\vec{a}}{\|\vec{n}\|}=\dfrac{1}{\sqrt{(1/2)^2+1^2}}=2/\sqrt{5}$.

(2c) $3x+y=2$, normal vector is $\vec{m}=\binom{3}{1}$.

(2d) Angle between ℓ and m is the angle θ between their normals, whose cosine is $\cos\theta=\frac{\vec{n}\cdot\vec{m}}{\|\vec{n}\|\,\|\vec{m}\|}=\frac{-1/2}{\sqrt{5/4}\sqrt{10}}=-\frac{1}{\sqrt{50}}=-\frac{1}{10}\sqrt{2}$.

(3a) A possible normal vector is $\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-4\\4\\-4\end{pmatrix}$. Any (non zero) multiple of this vector is also a valid normal. The nicest would be $\frac{1}{4}\vec{n}=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$.

(3b) $\vec{n}\cdot(\vec{x}-\vec{a})=0$, or $\vec{n}\cdot\vec{x}=\vec{n}\cdot\vec{a}$. Using $\vec{n}$ and $\vec{a}$ from the first part we get $-4x_1+4x_2-4x_3=-8$. Here you could replace $\vec{a}$ by either $\vec{b}$ or $\vec{c}$. (Make sure you understand why; if you don't think about it, then ask someone).

(3c) Distance from D to $\mathcal{P}$ is $\frac{\vec{n}\cdot(\vec{d}-\vec{a})}{\|\vec{n}\|} = 4/\sqrt{3} = \frac{4}{3}\sqrt{3}$. There are many valid choices of normal $\vec{n}$ in part (i) of this problem, but they all give the same answer here.

Distance from O to $\mathcal{P}$ is $\frac{\vec{n}\cdot(\vec{0}-\vec{a})}{\|\vec{n}\|} = \frac{2}{3}\sqrt{3}$.

(3d) Since $\vec{n}\cdot(\vec{0}-\vec{a})$ and $\vec{n}\cdot(\vec{d}-\vec{a})$ have the same sign the point D and the origin lie on the same side of the plane $\mathcal{P}$.

(3e) The area of the triangle is $\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = 2\sqrt{3}$.

(3f) Intersection with x axis is A, the intersection with y-axis occurs at $(0,-2,0)$ and the intersection with the z-axis is B.

(4a) Since $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}-3\\1\\1\end{pmatrix}$ the plane through A, B, C has defining equation $-3x+y+z=3$. The coordinates $(2,1,3)$ of D do not satisfy this equation, so D is not on the plane ABC.

(4b) If E is on the plane through A, B, C then the coordinates of E satisfy the defining equation of this plane, so that $-3\cdot 1 + 1\cdot 1 + 1\cdot\alpha = 3$. This implies $\alpha = 5$.

(5) The linear approximation formula says that

$$\Delta f \approx f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y.$$

Use $(x_0,y_0) = (2,1)$ and $(\Delta x, \Delta y) = (0.01, -0.01)$ and you find that

$$\Delta f \approx (-2)(0.01) + (2)(-0.01) = -0.04.$$

Therefore $f(2.01, 0.99) = 4 + \Delta f \approx 4 - 0.04 = 3.96$.

(6) The equation for the tangent plane is

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

We have

$$x_0 = \sqrt{2}, \quad y_0 = 2, \quad z_0 = f(x_0,y_0) = 1$$

$$f_x(x_0,y_0) = 2x_0 e^{x_0^2-y_0} = 2\sqrt{2}$$

$$f_y(x_0,y_0) = -e^{x_0^2-y_0} = -1$$

so that the tangent plane has equation

$$z - 1 = 2\sqrt{2}(x-\sqrt{2}) + (-1)(y-2).$$

Simplification leads to

$$z = 2\sqrt{2}\,x - y - 1.$$

(7) We are asked to find

$$\frac{df(x(\theta), y(\theta))}{d\theta},$$

where

$$x(\theta) = \sqrt{2}\cos\theta, \qquad y(\theta) = \sqrt{2}\sin\theta.$$

The chain rule says that

$$\begin{aligned}\frac{df(x(\theta),y(\theta))}{d\theta}, &= f_x(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)\frac{dx}{d\theta}+f_y(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)\frac{dy}{d\theta}\\ &= f_x(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)\big(-\sqrt{2}\big)\sin\theta+f_y(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta)\sqrt{2}\cos\theta\end{aligned}$$

When $\theta=\pi/4$ we also have $\sin\theta=\cos\theta=\frac{1}{2}\sqrt{2}$, and thus

$$\begin{aligned}\frac{df(x(\theta),y(\theta))}{d\theta} &= f_x(1,1)\cdot\big(-\sqrt{2}\big)\cdot\tfrac{1}{2}\sqrt{2})+f_y(1,1)\cdot\sqrt{2}\cdot\tfrac{1}{2}\sqrt{2}\\ &= -4+12=8.\end{aligned}$$

(8a) The partial derivatives of g are given by

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial\sin(\pi f(x,y))}{\partial x}=\pi\cos(\pi f(x,y))\frac{\partial f}{\partial x},\\ \frac{\partial g}{\partial y} &= \pi\cos(\pi f(x,y))\frac{\partial f}{\partial y},\end{aligned}$$

so that

$$g_x(4,0)=\pi(\cos\pi)(2)=-2\pi,\quad g_y(4,0)=\pi(\cos\pi)(-3)=+3\pi.$$

(8b) The chain rule says that

$$\begin{aligned}h'(t) &= \frac{df(4\cos t,\sin t)}{dt}\\ &= f_x(4\cos t,\sin t)\frac{d4\cos t}{dt}+f_y(4\cos t,\sin t)\frac{d\sin t}{dt}\\ &= -4\sin(t)f_x(4\cos t,\sin t)+\cos(t)f_y(4\cos t,\sin t).\end{aligned}$$

Set $t=0$ to get

$$h'(0)=-4(0)f_x(4,0)+(1)f_y(4,0)=f_y(4,0)=-3.$$

(9) The linear approximation formula says

$$f(-0.98,1.01)=f(-1+\Delta x,1+\Delta y)\approx f(-1,1)+f_x(-1,1)\Delta x+f_y(-1,1)\Delta y,$$

where $\Delta x=0.02$ and $\Delta y=0.01$.

We know $f(-1,1)=2$, $f_x(-1,1)=2$, and $f_y(-1,1)=3$, so

$$f(-0.98,1.01)\approx 2+2\cdot 0.02+3\cdot 0.01=2.07.$$

(10) The equation for the tangent plane is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0).$$

where we have $x_0=1/6$, $y_0=1/4$,

$$z_0=f(x_0,y_0)=\sin\big(-\pi/4-3\pi/6\big)=-\sin\frac{3}{4}\pi=-\frac{1}{2}\sqrt{2}.$$

The derivatives are

$$f_x(x_0, y_0) = -\pi\cos\bigl(-\pi/4 - 3\pi/6\bigr) = -\frac{\pi}{2}\sqrt{2}$$
$$f_y(x_0, y_0) = -3\pi\cos\bigl(-\pi/4 - 3\pi/6\bigr) = -\frac{3\pi}{2}\sqrt{2}.$$

Therefore the equation for the tangent plane is

$$z + \frac{1}{2}\sqrt{2} = -\frac{\pi}{2}\sqrt{2}(x - \tfrac{1}{4}) - \frac{3\pi}{2}\sqrt{2}(y - \tfrac{1}{6}).$$

Simplification leads to

$$z = -\frac{\pi}{2}\sqrt{2}x - \frac{3\pi}{2}\sqrt{2}y - \frac{1}{2}\sqrt{2} + \frac{3}{8}\pi\sqrt{2}.$$

(11) For any value of t you have

$$\begin{aligned} g'(t) &= \frac{\mathrm{d}f(t^2, 4t^4 + t^3 - t)}{\mathrm{d}t} \\ &= f_x(t^2, 4t^4 + t^3 - t)\frac{\mathrm{d}t^2}{\mathrm{d}t} + f_y(t^2, 4t^4 + t^3 - t)\frac{\mathrm{d}\ 4t^4 + t^3 - t}{\mathrm{d}t} \\ &= 2tf_x(t^2, 4t^4 + t^3 - t) + (16t^3 + 3t^2 - 1)f_y(t^2, 4t^4 + t^3 - t) \end{aligned}$$

If you set $t = 1$ you get

$$\begin{aligned} g'(1) &= 2\cdot 1\cdot f_x(1,4) + (16\cdot 1^3 + 3\cdot 1^2 - 1)f_y(1,4) \\ &= 2A + 18B \end{aligned}$$

For $t = -1$ you get

$$\begin{aligned} g'(-1) &= 2\cdot(-1)\cdot f_x(1,4) + (16\cdot(-1)^3 + 3\cdot(-1)^2 - 1)f_y(1,4) \\ &= -2A - 14B \end{aligned}$$

(12a) First compute the partial derivatives of f at $(1, 3)$. They are

$$f_x = \frac{2x}{1 + x^2 + y^2}$$
$$f_y = \frac{2y}{1 + x^2 + y^2}$$

so that

$$f_x(1,3) = \frac{2}{1 + 1^2 + 3^2} = \frac{2}{11}$$
$$f_y(1,3) = \frac{2\cdot 3}{1 + 1^2 + 3^2} = \frac{6}{11}$$

Therefore the equation of the tangent plane at (x_0, y_0, z_0) is

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

i.e.

$$z - \ln 11 = \frac{2}{11}(x - 1) + \frac{6}{11}(y - 3).$$

(12b) Simply set $x = y = 0$ in the equation we found in part (i), to get

$$z = \ln 11 + \frac{2}{11}(-1) + \frac{6}{11}(-3) \quad \left(= \ln 11 - \frac{20}{11}\right).$$

(13a) $f_x = \dfrac{2}{1+y^2}$, $\quad f_y = \dfrac{-4xy}{(1+y^2)^2}$, $\quad \vec{\nabla} f = \begin{pmatrix} 2/(1+y^2) \\ -4xy/(1+y^2)^2 \end{pmatrix}$

(13b) The equation for the tangent plane to the graph of any function is

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Since $z_0 = f(x_0, y_0) = \frac{2 \cdot 3}{1+1^2} = 3$, $f_x(x_0, y_0) = \frac{2}{1+1^2} = 1$, and $f_y(x_0, y_0) = -\frac{4 \cdot 3 \cdot 1}{(1+1^2)^2} = -3$, the equation for this tangent plane is

$$z = 3 + 1 \cdot (x - 3) + (-3) \cdot (y - 1) = x - 3y + 3.$$

(13c) The linear approximation formula says

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

Hence

$$f(3.1, 0.98) \approx f(3, 1) + 1 \cdot (0.1) + (-3) \cdot (-0.02) = 3 + 0.1 + 3 \cdot 0.02 = 3.16$$

(14a) The chain rule says

$$\begin{aligned} h'(t) &= \frac{df(2\cos t, \sin 3t)}{dt} \\ &= f_x(2\cos t, \sin 3t)\frac{d2\cos t}{dt} + f_y(2\cos t, \sin 3t)\frac{d\sin 3t}{dt} \\ &= -2(\sin t)\ f_x(2\cos t, \sin 3t) + 3(\cos 3t)\ f_y(2\cos t, \sin 3t) \end{aligned}$$

At $t = 0$ this becomes

$$h'(0) = -2 \cdot 0 \cdot f_x(2, 0) + 3 \cdot 1 \cdot f_y(2, 0) = -6.$$

(14b) The gradient $\vec{\nabla} f(x_0, y_0)$ is perpendicular to the tangent to the level set of f at the point (x_0, y_0). Thus $\binom{2}{-1}$ is a normal to the tangent of the level set at $(0, 3)$.

We know that the equation of the line with normal $\vec{n}$ through the point with position vector $\vec{a}$ is $\vec{n} \cdot (\vec{x} - \vec{a}) = 0$, so the equation for the tangent line is

$$2\big(x - 0\big) + (-1)\big(y - 3\big) = 0.$$

(14c) To decrease a function you must go in the direction opposite to its gradient, so we must find $\vec{\nabla} g(x, y)$ if we know $x = 2$, $y = 1$ and $g(x, y) = 1$. Since g satisfies the equation $f(x, y, g(x, y)) = C$ for some constant C, its derivatives are given by the implicit function theorem (or, you can find them by implicit differentiation). They are

$$\frac{\partial g}{\partial x}(2, 1) = -\frac{f_x(2, 1, 1)}{f_z(2, 1, 1)} = -\frac{6}{8}, \text{ and } \frac{\partial g}{\partial y}(2, 1) = -\frac{f_y(2, 1, 1)}{f_z(2, 1, 1)} = -\frac{12}{8},$$

Hence we must move in the direction of the vector

$$-\vec{\boldsymbol{\nabla}} g(2,1) = \begin{pmatrix} 6/8 \\ 12/8 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 3/2 \end{pmatrix}.$$

Any positive multiple of this vector (such as $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$) also points in the right direction.