MATH 221 — SECOND MIDTERM **November 7, 2007**

(1) Compute the derivative of each of the following (show your work).

(a) $f(x) = \dfrac{\sin x}{1 + \cos 2x} \implies f'(x) = \dots$

(b) $y = \ln \dfrac{1 + x^2}{(1 - x)^2} \implies \dfrac{dy}{dx} = \dots$

(c) $f(x) = \sqrt{1 + e^{-x}} \implies f'(x) =$

(d) Find the 10th derivative of $f(x) = e^{2x} - e^{-x/2}$.

(2) The function $y = f(x)$ satisfies

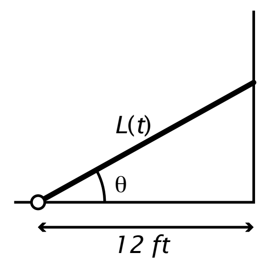
$$e^y + y = xy/2$$

for all x.

(a) For which value of x does one have $y = 2$?

(b) Compute $\dfrac{dy}{dx}$ when $y = 2$.

(3) An extending ladder is placed against a wall. The angle it makes with the floor is θ and its length is L. The bottom of the ladder is anchored at a point which is 12 feet away from the wall.
When the ladder is 15 feet long, its length is decreasing by 0.5ft/sec. What is the rate of change of the angle θ at that moment?



(4) Consider the function $f(x) = \sqrt{x} - \sqrt[4]{x}$

(a) Find the interval(s) on which f is increasing. Find the local maxima and minima of f on the interval $0 \le x < \infty$.

(b) Find the intervals on which f is convex or concave. Find the inflection point(s) in the graph of f.

(5) Find the **largest value** and the **smallest value** which the function

$$f(x) = \arcsin(x) + 2\sqrt{1 - x^2}$$

can have on the (closed) interval $0 \le x \le 1$.

Answers and comments.

(1) Compute the derivative of each of the following (show your work).

(a) $f(x) = \dfrac{\sin x}{1+\cos 2x} \implies f'(x) = \dots$

Solution:

$$f'(x) = \frac{\cos x \cdot (1+\cos 2x) - \sin x \cdot (-2\sin 2x)}{(1+\cos 2x)^2} = \frac{\cos x(1+\cos 2x) + 2\sin x \sin 2x}{(1+\cos 2x)^2}$$

(b) $y = \ln \dfrac{1+x^2}{(1-x)^2} \implies \dfrac{dy}{dx} = \dots$

Solution: Simplify the function first using properties of the logarithm!

$$f(x) = \ln(1+x^2) - 2\ln(1-x).$$

Then differentiate:

$$f'(x) = \frac{2x}{1+x^2} + \frac{2}{1-x}$$

If you don't first simplify, then you can apply chain rule, quotient rule, etc. and you should get the same answer, but it's much more work.

(c) $f(x) = \sqrt{1+e^{-x}} \implies f'(x) = \dots$

$$f'(x) = \frac{1}{2}\big(1+e^{-x}\big)^{-1/2} \cdot (-e^{-x}) = \frac{-e^{-x}}{2\sqrt{1+e^{-x}}}$$

(d) Find the 10th derivative of $f(x) = e^{2x} - e^{-x/2}$.

Solution: Every time you differentiate e^{2x} you get $2e^{2x}$, so after differentiating ten times you get $2^{10}e^{2x}$. Similarly differentiating $e^{-x/2}$ ten times gives you $(-1/2)^{10}e^{-x/2} = \frac{1}{2^{10}}e^{-x/2}$. The answer is therefore

$$f^{(10)}(x) = 2^{10}e^{2x} - 2^{-10}e^{-x/2}.$$

(2) The function $y = f(x)$ satisfies

$$e^y + y = xy/2$$

for all x.

(a) For which value of x does one have $y = 2$?

Solution: If $y = 2$ then we get the following equation for x

$$e^2 + 2 = x \cdot 2/2 = x,$$

so $x = e^2 + 2$.

(b) Compute $\dfrac{dy}{dx}$ when $y = 2$.

Solution: We use the method of implicit differentiation. Differentiate the equation to get

$$\frac{d(e^y + y)}{dx} = \frac{d(xy/2)}{dx} \implies (e^y + 1)\frac{dy}{dx} = \frac{y}{2} + \frac{x}{2}\frac{dy}{dx}.$$

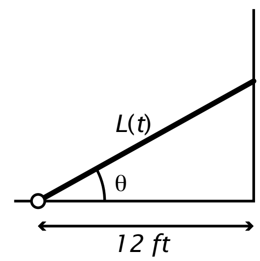
Solve this for dy/dx:

$$\frac{dy}{dx} = \frac{y/2}{e^y + 1 - x/2}.$$

To find the derivative when $y = 2$ we need to know which value x has when $y = 1$. This is the same question as (a) where we found that when $y = 2$ you have $x = e^2 + 2$. Therefore

$$\left(\frac{dy}{dx}\right)_{\text{when } y=1} = \frac{1}{e^2 + 1 - (e^2+2)/2} = 2e^{-2}.$$

(3) An extending ladder is placed against a wall. The angle it makes with the floor is θ and its length is L. The bottom of the ladder is anchored at a point which is 12 feet away from the wall.
When the ladder is 15 feet long, its length is decreasing by 0.5ft/sec. What is the rate of change of the angle θ at that moment?



Solution: The length of the ladder and the angle θ are related by

$$L(t)\cos\theta(t) = 12\text{ft}.$$

Differentiate to get

$$L'(t)\cos\theta(t) - L(t)\sin\theta(t)\cdot\theta'(t) = 0.$$

We are given that $L' = -0.5$, $L = 15$, and hence also $\cos\theta = \frac{12}{15} = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$.
Therefore

$$\theta'(t) = \frac{L'(t)\cos\theta(t)}{L(t)\sin\theta(t)} = \frac{-0.5\cdot\frac{4}{5}}{15\cdot\frac{3}{5}} = -\frac{2}{45}\text{rad/sec.}$$

(4) Consider the function $f(x) = \sqrt{x} - \sqrt[4]{x}$
 (a) Find the interval(s) on which f is increasing. Find the local maxima and minima of f on the interval $0 \le x < \infty$.
 (b) Find the intervals on which f is convex or concave. Find the inflection point(s) in the graph of f.

Solution: (a) We are only asked to find local maxima and minima in the interval $0 \le x < \infty$. If you look at the function you see that it isn't defined for $x < 0$.
We compute the derivative

$$f'(x) = \tfrac{1}{2}x^{-1/2} - \tfrac{1}{4}x^{-3/4}$$

and factor the result:

$$f'(x) = x^{-3/4}(\tfrac{1}{2}x^{1/4} - \tfrac{1}{4}) = \tfrac{1}{2}x^{-3/4}(x^{1/4} - \tfrac{1}{2}).$$

The first factors ($\frac{1}{2}$ and $x^{-3/4}$) are always positive, the factor $x^{1/4} - \frac{1}{2} = \sqrt[4]{x} - \frac{1}{2}$ is positive for $x > (\frac{1}{2})^4 = \frac{1}{16}$ and negative for $0 < x < (\frac{1}{2})^4 = \frac{1}{16}$.

So $f(x)$ is decreasing for $0 < x < \frac{1}{16}$ and increasing for $x > \frac{1}{16}$. The function has a global minimum at $x = \frac{1}{16}$.

(b) To find any inflection points we compute the second derivative of f

$$f''(x) = -\tfrac{1}{4}x^{-3/2} + \tfrac{3}{16}x^{-7/4}$$

and factor it,

$$f''(x) = x^{-7/4}\bigl(-\tfrac{1}{4}x^{1/4} + \tfrac{3}{16}\bigr) = -\tfrac{1}{4}x^{-7/4}\bigl(x^{1/4} - \tfrac{3}{4}\bigr)$$

So $f''(x) > 0$ and the graph is convex when $0 < x < (\frac{3}{4})^4$, while $f''(x) < 0$ and the graph is concave when $x > (\frac{3}{4})^4$. There is an inflection point at $x = (\frac{3}{4})^4$.

(5) Find the **largest value** and the **smallest value** which the function

$$f(x) = \arcsin(x) + 2\sqrt{1 - x^2}$$

can have on the (closed) interval $0 \le x \le 1$.

Solution: The derivative of this function is

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + 2\,\frac{-2x}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + \frac{-2x}{\sqrt{1-x^2}} = \frac{1-2x}{\sqrt{1-x^2}}$$

So the function is increasing for $0 < x < \frac{1}{2}$ and decreasing for $\frac{1}{2} < x < 1$. It has a global maximum at $x = \frac{1}{2}$, and the largest value of $f(x)$ is therefore

$$f(\tfrac{1}{2}) = \arcsin(\tfrac{1}{2}) + 2\sqrt{1 - (\tfrac{1}{2})^2} = \frac{\pi}{6} + \sqrt{3}.$$

The function has local minima at the end points $x = 0$ and $x = 1$. To see which is the global minimum you compute the function values

$$f(0) = \arcsin(0) + 2\sqrt{1 - 0^2} = 0 + 2 \cdot 1 = 2,$$
$$f(1) = \arcsin(1) + 2\sqrt{1 - 1^2} = \frac{\pi}{2} + 2 \cdot 0 = \frac{\pi}{2}.$$

Since $2 > \frac{\pi}{2}$ the largest value of $f(x)$ is achieved when $x = 0$, and it is $f(0) = 2$.