Math 221 (2008fall) second midterm – solutions and comments

1. (30 pts) Compute these derivatives

(a) $f(x) = \dfrac{2+x^2}{(1+x)^2} \implies f'(x) = \dfrac{2x(1+x)^2 - (2+x^2)2(1+x)}{(1+x)^4} = \dfrac{2x-4}{(1+x)^3}$.

(b) $\dfrac{\mathrm{d}\big(\arcsin\sqrt{x}\big)}{\mathrm{d}x} = \dfrac{1}{\sqrt{1-x}}\dfrac{1}{2\sqrt{x}}$.

(c) $\dfrac{\mathrm{d}\ln\big(2e^{\cos x}\big)}{\mathrm{d}x} =?$

Solution: Note that $\ln(2e^{\cos x}) = \ln(2) + \ln e^{\cos x} = \ln(2) + \cos x$. So the derivative of this function is

$$\frac{\mathrm{d}\ln\big(2e^{\cos x}\big)}{\mathrm{d}x} = \frac{\mathrm{d}(\ln(2)+\cos x)}{\mathrm{d}x} = -\sin x.$$

(d) $g(x) = \sqrt[4]{e^{2x}+\cos x} \implies g'(x) = \dfrac{1}{4}\big(e^{2x}+\cos x\big)^{-3/4}\big(2e^{2x} - \sin x\big)$.

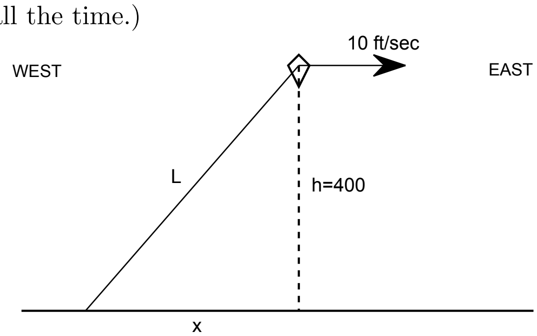
(e) Compute the 10^{th} derivative of $f(x) = \sin(2x) + \dfrac{1}{1-x}$.

Solution: $f^{(10)}(x) = -2^{10}\sin(2x) + \dfrac{1\cdot 2\cdot 3\cdots 9\cdot 10}{(1-x)^{11}}$

Grader's comments:

In (e) a common source of trouble was that students used the quotient rule to compute the derivatives of $1/(1-x)$. This is not wrong, but unless you simplify after each application of the quotient rule your will get very ugly expressions. A simpler way to compute the derivatives is to use (negative) exponents and write $1/(1-x) = (1-x)^{-1}$.

2. (15 pts) I am flying a kite in a wind blowing toward the east. The kite is moving horizontally at an altitude of 400 feet, and so I must pay out string from the spool that I am holding. Find the rate at which the string is unwinding from the spool when the kite is directly over the point 300 feet east of where I am standing, if at that moment the kite is moving east at 10 feet per second (and staying at the same altitude all the time.)



Solution: Several solutions are possible. Call the length of the string $L(t)$ and the horizontal distance to the kite $x(t)$. We are given that $\frac{\mathrm{d}x}{\mathrm{d}t} = 10$.

By Pythogoras we have $L^2 = (400)^2 + x^2$. At any time we therefore have

$$2L\frac{\mathrm{d}L}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t}, \text{ so } \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{x}{L}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

At the moment when $x = 400$ we have $L = \sqrt{(400)^2 + (300)^2} = 500$, and thus

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{300}{500}\,10 = 6\text{ft/sec}.$$

3. (15 pts) A function $y = f(x)$ satisfies the equation

$$2x + \sin(x) = 3y + y^3.$$

(a) Which of the following statements are true (you must of course explain your answer):

$$f(0) = 0? \qquad\qquad f(\pi) = 2?$$

Solution: How much is $y = f(0)$? To find y we have to solve

$$2 \cdot 0 + \sin(0) = 3y + y^3, \text{ i.e. } 3y + y^3 = 0$$

for y. Factor the right hand side to find

$$y(3 + y^2) = 0,$$

Since $3 + y^2 \neq 0$ always, the only possible solution is $y = 0$.
How much is $y = f(\pi)$? It is the solution of

$$2\pi + \sin\pi = 3y + y^3.$$

By substituting $x = \pi, y = 2$ you get $2\pi + \sin(\pi) = 3 \cdot 2 + 2^3$, which is not true. Therefore $(x, y) = (\pi, 2)$ is not on the graph of $y = f(x)$, i.e. $f(\pi) = 2$ is not true.

Grader's comments: 1. If $y = f(x)$ is defined implicitly by an expression in x and y, showing that $f(a) = b$ requires more work than just plugging $x = a$ and $y = b$ into the expression, and seeing if the result is a true equation. In part (a), to show that $f(0) = 0$, one must show that the ONLY solution to $y = f(0)$ is $y = 0$.
For example, if the same question were asked about a function $y = f(x)$ which satisfies

$$x + \sin\pi x = 3y - y^3$$

then there are three possibilities for $y = f(0)$, namely the three solutions of $3y - y^3 = 0$ (they are $y = 0$, $y = \pm\sqrt{3}$.)
2. In contrast, to show that $f(a) \neq b$, one only needs to plug $x = a$ and $y = b$ into the expression and show that the resulting equation is not true. So, to show that $f(\pi) = 2$ is not true, one can just plug in.

(b) Compute $f'(0)$.
Solution: Differentiate the implicit equation for $y = f(x)$:

$$2 + \cos x = (3 + 3y^2)\frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 + \cos x}{3 + 3y^2}$$

Since $f(0) = 0$ we get

$$f'(0) = \frac{2 + \cos 0}{3 + 3 \cdot 0^2} = 1.$$

(c) Can $f'(x)$ ever be negative? Explain your answer.
Solution: The formula for $\mathrm{d}y/\mathrm{d}x$ in part (b) tells us that we always have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 + \cos x}{3 + 3y^2}$$

Since $2 + \cos x \geq 2 - 1 = 1$ and since $3 + 3y^2 \geq 3$ this fraction is always positive, so $f'(x)$ will always be positive.

4. (25 pts) Consider the function $f(x) = \dfrac{x^2}{4 + x^2}$.

(a) Find the intervals on which f is increasing or decreasing.
Solution: $f'(x) = \frac{8x}{(4+x^2)^2}$, so $f(x)$ is increasing for $x > 0$ and decreasing for $x < 0$.

(b) Find the local maxima and minima of f. Which of these are global maxima or minima?
Solution: In part (a) we found that $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$. Therefore $f(0)$ is smaller than $f(x)$ for all other values of x, and it follows that $f(x)$ has a global minimum at $x = 0$.
There are no other stationary points, so $f(x)$ has no other local maxima or minima, since any local maximum or minimum must be a stationary point.

(c) Find the inflection points on the graph of f, and the intervals where the function is convex and concave.
Solution: The second derivative is

$$f''(x) = 8\frac{4-3x^2}{(4+x^2)^3}.$$

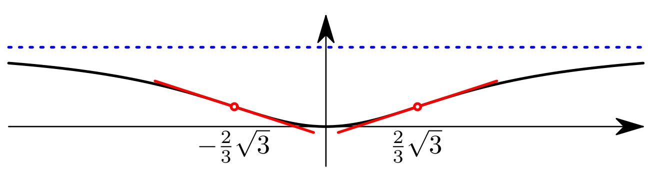
One has $f''(x) > 0$ for $-\frac{2}{3}\sqrt{3} < x < +\frac{2}{3}\sqrt{3}$, so the graph is convex (curved upwards) for these values of x. In the intervals $x < -\frac{2}{3}\sqrt{3}$ and $x > \frac{2}{3}\sqrt{3}$ one has $f''(x) > 0$, so the graph is concave there.
There are two inflection points, namely at $x = \pm\frac{2}{3}\sqrt{3}$.

(d) Find all horizontal and vertical asymptotes of the graph of f.
Solution: The function is continuous everywhere so it has no vertical asymptotes.
To find horizontal asymptotes we compute

$$\lim_{x\to\infty} \frac{x^2}{4+x^2} = \lim_{x\to\infty} \frac{1}{\frac{4}{x^2}+1} = 1,$$

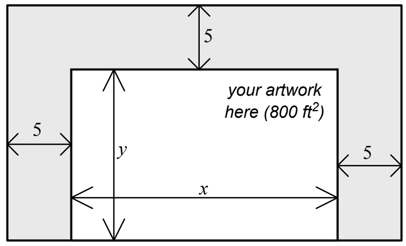
The limit for $x \to -\infty$ has the same value and is computed in the same way.
Therefore the line $y = 1$ is a horizontal asymptote for the graph of the function f both for $x \to \infty$ and for $x \to -\infty$.

(e) Sketch the graph.



Grader's comments: – Points were lost for forgetting to explain why there is no local maximum, or why a $x = 0$ is a local or global minimum.
– The graph you drew in (e) should be consistent with all claims you made in your answers (a). . . (d).
– A common source of problems for many students was in the calculation of $f''(x)$, and in finding sign changes of this second derivative. It is almost never a good idea to expand powers like $(4+x^2)^2$, or multiply polynomials, especially if you later have to factor what you're computing.

5. (15 pts) You are designing a mural and you would like to have a margin of 5 feet on the left, right and top of the artwork, but none on the bottom. If you allow 800 ft^2 for the area containing the artwork itself, what dimensions should the wall have if we want to minimize the total area of the wall (i.e. of the artwork and the margins.)
(First make a drawing!)
Solution: Several solutions are possible, depending on which lengths you call x and y. Here is one:



Let the width and height of the art work be x and y. Then $xy = 800$, so that $y = \frac{800}{x}$. Any positive width is allowed (meaning: for any width $x > 0$ you can make a mural with 800ft^2 artwork and a 5ft margin around it.)

The total area needed for the wall is $(x + 10)(y + 5)$, so we want to minimize

$$A(x) = (x + 10)(\frac{800}{x} + 5) = 5x + 850 + \frac{8000}{x}, \qquad 0 < x < \infty.$$

The derivative is

$$A'(x) = 5 - \frac{8000}{x^2}.$$

Therefore there is one critical point, namely at $x = \sqrt{1600} = 40$.

For $x < 40$ the function is decreasing ($A'(x) < 0$), and for $x > 40$ the function $A(x)$ is increasing again. Thus the area $A(x)$ is minimal when $x = 40$, and $y = \frac{800}{40} = 20$.

The smallest wall will be 50ft wide and 25ft high.