MATH 221 — THE FIRST MIDTERM

October 2, 2007

(1) (a) If one defines a function by saying that

$$y = f(x) \iff y \text{ is the largest solution of } y^2 + 6y = x$$

then what is the domain of f, and find a formula for $f(x)$.
Solution: For any given x the equation $y^2 + 6y = x$ has at most two solutions. They are given by the quadratic formula, and the largest of the two solutions is

$$y = \frac{-6 + \sqrt{36 + 4x}}{2}.$$

This solution exists if $36 + 4x \geq 0$, so the domain consists of all x with $x \geq -9$.
Answer: The function is

$$f(x) = -3 + \frac{1}{2}\sqrt{36 + 4x} = -3 + \sqrt{9 + x}$$

and its domain is $[-9, \infty)$.

(b) Does there exist a function f such that $f(2x+4) = x \sin x$ holds for all real numbers x? If there is such a function, then find the domain of f and a formula for $f(x)$.
Solution: How much would "f of anything" be? We know that $f(2x + 4) = x \sin x$ for any number x. Call $u = 2x + 4$. Then for any u you canalways find x from $x = \frac{u-4}{2}$. What we are given is then that

$$f(u) = \frac{u-4}{2} \sin \frac{u-4}{2}$$

is always true.
Answer: Yes such a function f exists, it is defined for all real numbers, and it is given by the above formula.

(c) Does there exist a function f such that $f(x^2 + 4) = x$ holds for all real numbers x? If there is such a function, then find the domain of f and a formula for $f(x)$.
Answer: No, f does not exist, because the equation $f(x^2 + 4) = x$ would have to be true for both $x = -1$ and $x = 1$, which would give us

$$f(5) = f\left((-1)^2 + 4\right) = -1 \text{ and } f(5) = f\left((+1)^2 + 5\right) = +1.$$

The function value $f(5)$ can't be $+1$ and -1 at the same time.

(2) (a) State the ε–δ definition of "$\lim_{x\to a} f(x) = L$".
Answer: See the lecture notes.

(b) Show, using the ε–δ definition, that $\displaystyle\lim_{x\to 2} \frac{12}{x+2} = 3$.
Solution: For any given $\varepsilon > 0$ we have to find a δ such that $|\frac{12}{x+2} - 3| < \varepsilon$ holds whenever $|x - 2| < \delta$.
We have

$$\left|\frac{12}{x+2} - 3\right| = \left|\frac{6 - 3x}{x+2}\right| = \frac{3}{|x+2|}|x - 2|.$$

If we agree never to choose $\delta > 1$, then $|x - 2| < \delta$ implies $|x - 3| < 1$ and hence $1 < x < 3$. Therefore any x which satisfies $|x - 3| < \delta$ will satisfy

$$\frac{3}{|x+2|} < \frac{3}{1+2} = 1.$$

Hence one will also have

$$|\frac{12}{x+2} - 3| < |x - 2|.$$

If we want to be sure that $|\frac{12}{x+2} - 3| < \varepsilon$ then we should require $|x - 2| < \varepsilon$. We can therefore choose any $\delta \leq \varepsilon$. To keep our promise that we will never choose $\delta > 1$, we decide to choose

$$\delta = \min\{1, \varepsilon\}.$$

(3) (a) Compute the following limits using the limit properties:

$$\lim_{x\to 2} \frac{x - \frac{4}{x}}{x^2 - 2x} =$$ *Answer:* The limit is 1.

$$\lim_{x\to 0} \frac{\sin \pi x}{\tan 3x} =$$ *Answer:* $\frac{\pi}{3}$.

$$\lim_{t\to 0} \frac{2 - \sqrt{4 + t^2}}{t^2} =$$ *Answer:* $\frac{-1}{4}$.

(b) Show that the limit properties imply that the limit

$$\lim_{x\to 0} \frac{x+1}{x(x+3)}$$

does not exist.

Solution: Assume the limit exists. Give it a name,

$$L = \lim_{x\to 0} \frac{x+1}{x(x+3)}.$$

Then multiply the given limit with

$$0 = \lim_{x\to 0} x$$

You get on one hand

$$\begin{aligned} 0 &= 0 \cdot L \\ &= \big(\lim_{x\to 0} x\big)\big(\lim_{x\to 0} \frac{x+1}{x(x+3)}\big) \\ &= \lim_{x\to 0} x \cdot \frac{x+1}{x(x+3)} \\ &= \lim_{x\to 0} \frac{x+1}{x+3} \\ &= \frac{1}{3}. \end{aligned}$$

So we find $0 = \frac{1}{3}$ which can't be true. Therefore our assumption that the limit L existed must have been false.

(4) For which values of the constants a and b is the function

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ a + x^2 & \text{for } 0 < x \le 2 \\ b/x & \text{for } x > 2 \end{cases}$$

continuous?

Answer: $a = 0$, $b = 8$.

(5) Use the definition of the derivative as a limit to find

(a) $f'(1)$ if $f(x) = \sqrt{x+3}$. *Answer:* 1/4.

(b) $g'(2)$ if $g(x) = x^2 + \dfrac{2}{x}$. *Answer:* $3\frac{1}{2}$.