Answers and Hints

(October 30, 2008)

1 The decimal expansion of

$$1/7 = 0.\overline{142857}\,142857\,142857\,\cdots$$

repeats after 6 digits. Since $2007 = 334 \times 6 + 3$ the 2007^{th} digit is the same as the 3^{rd}, which happens to be a 2.

3 $100x = 31.313131\cdots = 31 + x \implies 99x = 31 \implies x = \frac{31}{99}$.
Similarly, $y = \frac{273}{999}$.
In z the initial "2" is not part of the repeating pattern, so subtract it: $z = 0.2 + 0.0154154154\cdots$. Now

$$\begin{aligned} 1000\times 0.0154154154\cdots &= 15.4154154154\cdots \\ &= 15.4 + 0.0154154154\cdots \\ &= 15\tfrac{2}{5} + 0.0154154154\cdots \\ \implies 0.0154154\cdots &= \frac{15\frac{2}{5}}{999}. \end{aligned}$$

From this you get

$$z = \tfrac{1}{5} + \frac{15\frac{2}{5}}{999} = \frac{1076}{4995}.$$

5 $\mathcal{A} = [1, 2]$ contains infinitely many points.
$\mathcal{C} = (-\infty, \frac{3}{2} - \frac{1}{2}\sqrt{21}) \cup (\frac{3}{2} + \frac{1}{2}\sqrt{21}, \infty)$ contains infinitely many points.
$\mathcal{E} = \mathcal{A}$ contains infinitely many points.
$\mathcal{Q}$ consists of all solutions θ of $\sin\theta = \frac{1}{2}$. There are infinitely many solutions. They are $\pi/6$, $\pi/6 \pm 2\pi$, $\pi/6 \pm 4\pi$, …and are $5\pi/6$, $5\pi/6 \pm 2\pi$, $5\pi/6 \pm 4\pi$. A different way of saying this is: $\mathcal{Q}$ consists of the numbers

$$\frac{\pi}{6} + 2k\pi, \text{ and } \frac{5\pi}{6} + 2k\pi$$

where k is an arbitrary integer.

6 $\mathcal{A} \cap \mathcal{B}$ must always be an interval (or empty); $\mathcal{A} \cup \mathcal{B}$ does not have to be an interval, e.g. when $\mathcal{A} = (0, 1)$ and $\mathcal{B} = (2, 3)$.

8 They are the same function. Both are defined for all real numbers, and both will square whatever number you give them, so they are the same function.

9 Domain is all real numbers, $f(x) = 7/(1 + x^2)$.

10 Domain is $\{x \mid x \neq \pm 1\}$, $f(x) = 6/(x^2 - 1)$.

11 Domain is all reals, $f(x) = -x + 2|x|$.

14 Both are false:
(a) Since $\arcsin x$ is only defined if $-1 \le x \le 1$ and hence not for **all** x, it is not true that $\sin\bigl(\arcsin x\bigr) = x$ for all x. However, it is true that $\sin(\arcsin x) = x$ for all x in the interval $[-1, 1]$.
(b) $\arcsin(\sin x)$ is defined for all x since $\sin x$ is defined for all x, and $\sin x$ is always between -1 and 1. However the arcsine function always returns a number (angle) between $-\pi/2$ and $\pi/2$, so $\arcsin(\sin x) = x$ can't be true when $x > \pi/2$ or $x < -\pi/2$.

19 **(a)** $f(0) = 9/4$. **(e)** $f(f(2)) = (\frac{\frac{1}{4} - 3}{2})^2 = \cdots$
Domain is all reals; Range is $[0, \infty)$.

21 No, there is no such function.

23 Range of f is $[3, \infty)$. Range of k is $[-3, 5]$. Range of ℓ is $(0, 1]$ (Note that 0 is not included).

24 **(c)** (x, y) lies on one (or more) of the lines if and only $y \ge -x^2/4$.

26 The graphs of f and g do not intersect if $n^2 + 4m < 0$.

30 **(a)**

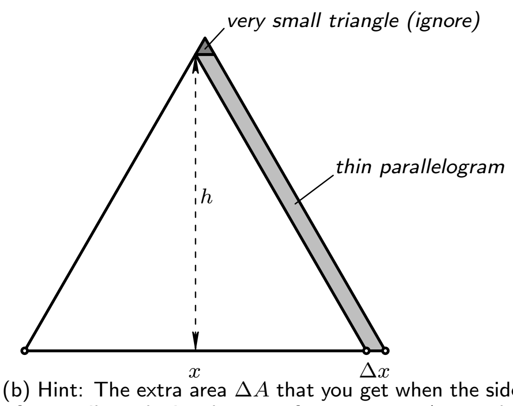
$$\begin{aligned} \Delta y &= (x + \Delta x)^2 - 2(x + \Delta x) + 1 - [x^2 - 2x + 1] \\ &= (2x - 2)\Delta x + (\Delta x)^2 \text{ so that} \\ \frac{\Delta y}{\Delta x} &= 2x - 2 + \Delta x \end{aligned}$$

31 At A and B the graph of f is tangent to the drawn lines, so the derivative at A is -1 and ther derivative at B is $+1$.

32 Δx : feet. Δy pounds. $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$ are measured in pounds per feet.

33 Gallons per second.

34 (a) $A(x)$ is an area so it has units square inch and x is measured in inches, so $\dfrac{dA}{dx}$ is measured in $\dfrac{\text{inch}^2}{\text{inch}} = \text{inch}$.



(b) Hint: The extra area ΔA that you get when the side of an equilateral triangle grows from x to $x + \Delta x$ can be split into a thin parallelogram and a very tiny triangle. Ignore the area of the tiny triangle since the area of the parallelogram will be much larger. What is the area of this parallelogram?
The area of a parallelogram is "base time height" so here it is $h \times \Delta x$, where h is the height of the triangle.
Conclusion: $\dfrac{\Delta A}{\Delta x} \approx \dfrac{h\Delta x}{\Delta x} = h$.
The derivative is therefore the height of the triangle.

39 $\delta = \varepsilon/2$.

40 $\delta = \min\{1, \frac{1}{5}\varepsilon\}$

41 $|f(x) - (-7)| = |x^2 - 7x + 10| = |x - 2| \cdot |x - 5|$.
If you choose $\delta \le 1$ then $|x - 2| < \delta$ implies $1 < x < 3$, so that $|x - 5|$ is at most $|1 - 5| = 4$.
So, choosing $\delta \le 1$ we always have $|f(x) - L| < 4|x - 2|$ and $|f(x) - L| < \varepsilon$ will follow from $|x - 2| < \frac{1}{4}\varepsilon$.
Our choice is then: $\delta = \min\{1, \frac{1}{4}\varepsilon\}$.

42 $f(x) = x^3$, $a = 3$, $L = 27$.
When $x = 3$ one has $x^3 = 27$, so $x^3 - 27 = 0$ for $x = 3$. Therefore you can factor out $x - 3$ from $x^3 - 27$ by doing a long division. You get $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$, and thus

$$|f(x) - L| = |x^3 - 27| = |x^2 + 3x + 9| \cdot |x - 3|.$$

Never choose $\delta > 1$. Then $|x - 3| < \delta$ will imply $2 < x < 4$ and therefore

$$|x^2 + 3x + 9| \le 4^2 + 3 \cdot 4 + 9 = 37.$$

So if we always choose $\delta \le 1$, then we will always have

$$|x^3 - 27| \le 37\delta \quad \text{for} \quad |x - 3| < \delta.$$

Hence, if we choose $\delta = \min\left\{1, \frac{1}{37}\varepsilon\right\}$ then $|x - 3| < \delta$ guarantees $|x^3 - 27| < \varepsilon$.

44 $f(x) = \sqrt{x}$, $a = 4$, $L = 2$.
You have

$$\sqrt{x} - 2 = \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} + 2} = \frac{x - 4}{\sqrt{x} + 2}$$

and therefore

$$|f(x) - L| = \frac{1}{\sqrt{x} + 2}|x - 4|. \qquad (1)$$

Once again it would be nice if we could replace $1/(\sqrt{x} + 2)$ by a constant, and we achieve this by always choosing $\delta \le 1$. If we do that then for $|x - 4| < \delta$ we always have $3 < x < 5$ and hence

$$\frac{1}{\sqrt{x} + 2} < \frac{1}{\sqrt{3} + 2},$$

since $1/(\sqrt{x} + 2)$ increases as you decrease x.
So, if we always choose $\delta \le 1$ then $|x - 4| < \delta$ guarantees

$$|f(x) - 2| < \frac{1}{\sqrt{3} + 2}|x - 4|,$$

which prompts us to choose $\delta = \min\left\{1, (\sqrt{3} + 2)\varepsilon\right\}$.
A smarter solution: We **can** replace $1/(\sqrt{x} + 2)$ by a constant in (1), because for all x in the domain of f we have $\sqrt{x} \ge 0$, which implies

$$\frac{1}{\sqrt{x} + 2} \le \frac{1}{2}.$$

Therefore $|\sqrt{x} - 2| \le \frac{1}{2}|x - 4|$, and we could choose $\delta = 2\varepsilon$.

45 Hints:

$$\sqrt{x + 6} - 3 = \frac{x + 6 - 9}{\sqrt{x + 6} + 3} = \frac{x - 3}{\sqrt{x + 6} + 3}$$

so

$$|\sqrt{x + 6} - 3| \le \tfrac{1}{3}|x - 3|.$$

46 We have

$$\left|\frac{1 + x}{4 + x} - \frac{1}{2}\right| = \frac{1}{2}\left|\frac{x - 2}{4 + x}\right|.$$

If we choose $\delta \le 1$ then $|x - 2| < \delta$ implies $1 < x < 3$ so that

$$\frac{1}{7} < \frac{1}{4 + x} < \frac{1}{5}.$$

We don't care about the "$\frac{1}{7} < \cdots$" part, but the other inequality implies

$$\frac{1}{2}\left|\frac{x - 2}{4 + x}\right| < \tfrac{1}{10}|x - 2|.$$

So if we want $|f(x) - \frac{1}{2}| < \varepsilon$ then we must require $|x - 2| < 10\varepsilon$. This leads us to choose

$$\delta = \min\{1, 10\varepsilon\}.$$

51 The equation (**??**) already contains a function f, but that is not the right function. In (**??**) Δx is the variable, and $g(\Delta x) = (f(x + \Delta x) - f(x))/\Delta x$ is the function; we want $\lim_{\Delta x \to 0} g(\Delta x)$.

52 -9

53 Sneaky question: $\lim_{x \to 7-}(2x + 5) = \lim_{x \nearrow 7}(2x + 5) = 19$.

56 -1.

57 D.N.E.

58 $3/2$.

61 1.

62 DNE or $+\infty$.

64 $16/9$.

67 $A(\frac{2}{3}, -1)$; $B(\frac{2}{5}, 1)$; $C(\frac{2}{7}, -1)$; $D(-1, 0)$; $E(-\frac{2}{5}, -1)$.

68 False! The limit must not only exist *but also be equal to* $f(a)$!

69 There are of course many examples. Here are two: $f(x) = 1/x$ and $f(x) = \sin(\pi/x)$ (see §**??**)

70 False! Here's an example: $f(x) = \frac{1}{x}$ and $g(x) = x - \frac{1}{x}$. Then f and g don't have limits at $x = 0$, but $f(x) + g(x) = x$ *does* have a limit as $x \to 0$.

71 False again, as shown by the example $f(x) = g(x) = \frac{1}{x}$.

73 $1/6$.

74 $-1/4$.

75 $-1/(4\sqrt{2})$.

77 $\sin 2\alpha = 2\sin\alpha\cos\alpha$ so the limit is $\lim_{\alpha\to 0}\frac{2\sin\alpha\cos\alpha}{\sin\alpha} = \lim_{\alpha\to 0} 2\cos\alpha = 2$.

Other approach: $\dfrac{\sin 2\alpha}{\sin\alpha} = \dfrac{\frac{\sin 2\alpha}{2\alpha}}{\frac{\sin\alpha}{\alpha}}\cdot\dfrac{2\alpha}{\alpha}$. Take the limit and you get 2.

78 $\frac{3}{2}$.

79 Hint: $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Answer: the limit is 1.

80 $\frac{\tan 4\alpha}{\sin 2\alpha} = \frac{\tan 4\alpha}{4\alpha}\cdot\frac{2\alpha}{\sin 2\alpha}\cdot\frac{4\alpha}{2\alpha} = 1\cdot 1\cdot 2 = 2$

81 Hint: multiply top and bottom with $1+\cos x$.

82 Hint: substitute $\theta = \frac{\pi}{2} - \varphi$, and let $\varphi \to 0$. Answer: -1.

88 Substitute $\theta = x - \pi/2$ and remember that $\cos x = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$. You get

$$\lim_{x\to\pi/2}\frac{x-\frac{\pi}{2}}{\cos x} = \lim_{\theta\to 0}\frac{\theta}{-\sin\theta} = -1.$$

89 Similar to the previous problem, once you use $\tan x = \frac{\sin x}{\cos x}$. The answer is again -1.

91 Substitute $\theta = x - \pi$. Then $\lim_{x\to\pi}\theta = 0$, so

$$\lim_{x\to\pi}\frac{\sin x}{x-\pi} = \lim_{\theta\to 0}\frac{\sin(\pi+\theta)}{\theta} = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = -1.$$

Here you have to remember from trigonometry that $\sin(\pi+\theta) = -\sin\theta$.

93 Note that the limit is for $x\to\infty$! As x goes to infinity $\sin x$ oscillates up and down between -1 and $+1$. Dividing by x then gives you a quantity which goes to zero. To give a good proof you use the Sandwich Theorem like this:

Since $-1 \le \sin x \le 1$ for all x you have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Since both $-1/x$ and $1/x$ go to zero as $x\to\infty$ the function in the middle must also go to zero. Hence

$$\lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

97 No. As $x\to 0$ the quantity $\sin\frac{1}{x}$ oscillates between -1 and $+1$ and does not converge to any particular value. Therefore, no matter how you choose k, it will never be true that $\lim_{x\to 0}\sin\frac{1}{x} = k$, because the limit doesn't exist.

98 The function $f(x) = (\sin x)/x$ is continuous at all $x\ne 0$, so we only have to check that $\lim_{x\to 0} f(x) = f(0)$, i.e. $\lim_{x\to 0}\frac{\sin x}{2x} = A$. This only happens if you choose $A = \frac{1}{2}$.

109

$$f'(x) = 8x^7 + 24x^5 + 24x^3 + 8x$$

112 $f'(x) = \dfrac{-3x^4 + 8x^3 + 1}{x^8 + 2x^4 + 1}$

113 $f'(x) = 1$.

114 $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}$

115 $f'(x) = \dfrac{ad-bc}{(cx+d)^2} = \dfrac{ad-bc}{c^2x^2 + 2cdx + d^2}$

119 $f'(x) = \dfrac{2\sqrt{x}+1}{6\sqrt{x}(x+\sqrt{x})^{2/3}}$

120 $\phi'(t) = \dfrac{\sqrt{t}+2}{2t + 4\sqrt{t} + 2}$

121 $g'(s) = -\dfrac{\sqrt{s+1}}{\sqrt{1-s}(s^2+2s+1)}$

122 $h'(\rho) = \dfrac{2\sqrt{\rho}+1}{6\sqrt{\rho}(\rho+\sqrt{\rho})^{2/3}}$

123

$$(a) \qquad f'(x) = \frac{4}{3}x^{1/3}$$

$$(b) \qquad \frac{127^{4/3} - 125^{4/3}}{2} \approx f'(125) = \frac{4}{3}125^{1/3} = \frac{20}{3}$$

125

$$f' = -\frac{4x^5 + 8x^3 - 14x}{4x^8 + 28x^4 + 49}$$

$$g' = \frac{4x^5 + 8x^3 - 14x}{x^4 + 2x^2 + 1}$$

(b) FALSE ; (c) FALSE ; (d) TRUE

126

$$\frac{dx}{dt} = -\frac{4t}{t^4 + 2t^2 + 1}\ \frac{dy}{dt} = -\frac{2t^2 - 2}{t^4 + 2t^2 + 1}$$

$$u(x) = \frac{2t}{1-t^2}$$

$$\frac{du}{dt} = \frac{x'y - y'x}{y^2} = \frac{2t^2 + 2}{t^4 - 2t^2 + 1}$$

128 $f'(x) = 4(x+1)^3$
$f''(x) = 4\cdot 3(x+1)^2$
$f^{(3)}(x) = 4\cdot 3\cdot 2(x+1)$.
$g'(x) = 8x(x^2+1)^3$
$g''(x) = 8(x^2+1)^3 + 48x^2(x^2+1)^2$
$g'''(x) = 144x(x^2+1) + 192x^3(x^2+1)$.
$h'(x) = \frac{1}{2}(x-2)^{-1/2}$
$h''(x) = -\frac{1}{4}(x-2)^{-3/2}$
$h^{(3)}(x) = -\frac{3}{8}(x-2)^{-5/2}$.
$k'(x) = \frac{1}{3}(1+1/x^2)(x-1/x)^{-2/3}$
$k''(x) = -\frac{2}{3}x^{-3}(x-1/x)^{-2/3} - \frac{2}{9}(1+1/x^2)^2(x-1/x)^{-5/3}$
$k'''(x) = 2x^{-4}(x-1/x)^{-2/3} + \frac{8}{9}x^{-3}(1+1/x^2)(x-1/x)^{-5/3} + \frac{10}{27}(1+1/x^2)^3(x-1/x)^{-8/3}$

129 $f^{(10)}(x) = 12 \cdot 11 \cdot 10 \cdots 3x^2$.
$g^{(10)}(x) = 1 \cdot 2 \cdot 3 \cdots 10x^{-11}$
$h^{(10)}(x) = 12 \cdot 1 \cdot \cdot 2 \cdots 9 \cdot 10(1-x)^{-11}$
Long division leads to $k(x) = -x - 1 + \frac{1}{1-x}$ so $k^{(10)}(x) = 1 \cdot 2 \cdot 3 \cdots 9 \cdot 10(1-x)^{-11}$

134 $f'(x) = \cos x - \sin x$

135 $f'(x) = 2\cos x + 3\sin x$

137 $f'(x) = x\cos x$.

138 $f'(x) = x\sin x$.

139 $f'(x) = \big(x\cos x - \sin x\big)/x^2$

140 $f'(x) = -2\cos x \sin x$

141 Careful! If x is such that $\cos x > 0$, then the function is $f(x) = \cos x$ and $f'(x) = -\sin x$; on the other hand, if $\cos x < 0$ then $f(x) = -\cos x$, so that $f'(x) = +\sin x$.
The straightforward (unthinking) answer is $f'(x) = \frac{-\sin x \cos x}{\sqrt{1-\sin^2 x}}$, which is correct, but looks much more complicated than necessary.

142 $f'(x) = -\cos x(1+\sin x)^{-3/2}(1-\sin x)^{-1/2}$

143 $\cot'(x) = \frac{-1}{\sin^2 x} = -1 - \cot^2(x)$

144 To make the function continuous you need $a + b\pi/4 = \frac{1}{2}\sqrt{2}$. To make the function differentiable you need $b = -\frac{1}{2}\sqrt{2}$. Solve these equations for a and b and you find $a = \frac{1}{2}\sqrt{2}\big(1 + \frac{\pi}{4}\big)\big)$, $b = -\frac{1}{2}\sqrt{2}$.
See the next problem for a more detailed write up of the solution.

145 First we make sure that the function is continuous at $x = \pi/6$. We compute

$$f(\pi/6) = a + b\pi/6,$$
$$\lim_{x\nearrow\pi/6} f(x) = \tan\pi/6 = \tfrac{1}{2}\sqrt{3},$$
$$\lim_{x\searrow\pi/6} f(x) = a + b\pi/6.$$

These three quantities are equal if

$$a + b\pi/6 = \tfrac{1}{2}\sqrt{3}.$$

Assume from now on that a and b satisfy this condition. Then

$$f(\pi/6) = a + b\pi/6$$

but also

$$f(\pi/6) = \tfrac{1}{2}\sqrt{3}.$$

We will use this below.
To see if f is differentiable at $x = \pi/6$, we compute the left and right hand limits

$$R = \lim_{x\nearrow 0} \frac{f(x) - f(\pi/6)}{x - \pi/6} = \lim_{x\nearrow 0} \frac{\tan x - \tan\pi/6}{x - \pi/6},$$

and

$$L = \lim_{x\searrow 0} \frac{f(x) - f(\pi/6)}{x - \pi/6} = \lim_{x\searrow 0} \frac{a + bx - (a + b\pi/6)}{x - \pi/6}.$$

The limit R is by definition the derivative of the function $y = \tan x$ at $x = \pi/6$, so we know

$$R = \frac{1}{\cos^2\pi/6} = \frac{4}{3}.$$

The left hand limit is, again by definition, the derivative of the function $y = a + bx$ at $x = \pi/6$, which tells us that

$$L = b.$$

We want the left and right hand limits to be the same, so we get

$$b = \frac{4}{3}.$$

Continuity of the function told us that $a + b\pi/6 = \frac{1}{2}\sqrt{3}$, so we get

$$a = \tfrac{1}{2}\sqrt{3} - \frac{2\pi}{9}.$$

148 $f'(x) = 2\tan x/\cos^2 x$ and $f''(x) = 2/\cos^4 x + 4\tan x \sin x/\cos^3 x$.
Since $\tan^2 x = \frac{1}{\cos^2 x} - 1$ one has $g'(x) = f'(x)$ and $g''(x) = f''(x)$.

153 $f \circ g(x)$ is another way of expressing $f(g(x))$, so

$$v(x) = f \circ g(x) = \sqrt{1 + x^2},$$
$$w(x) = g \circ f(x) = 1 + (\sqrt{x})^2 = 1 + |x|.$$

Hence

$$v'(x) = \frac{x}{\sqrt{1 + x^2}}$$

and

$$w'(x) = \begin{cases} +1 & \text{when } x > 0 \\ -1 & \text{when } x < 0 \end{cases}$$

154 $f'(x) = 2\cos 2x + 3\sin 3x$.

155 $f'(x) = -\dfrac{\pi}{x^2}\cos\dfrac{\pi}{x}$

156
$$f'(x) = \cos(\cos 3x) \cdot (-\sin 3x) \cdot 3 = -3\sin(3x)\cos(\cos 3x).$$

157
$$f'(x) = \frac{x^2 \cdot 2x\cos x^2 - 2x\sin x^2}{x^4} = 2x^{-1}\cos x^2 - 2x^{-3}\sin x^2$$

158 $f'(x) = \dfrac{1}{\big(\cos\sqrt{1+x^2}\big)^2}\dfrac{1}{2\sqrt{1+x^2}} \cdot 2x$

159 $f'(x) = 2(\cos x)(-\sin x) + 2(\sin x^2) \cdot 2x$

161 $f'(x) = \cos\frac{\pi}{x} + \frac{\pi}{x}\sin\frac{\pi}{x}$. At C one has $x = -\frac{2}{3}$, so $\cos\frac{\pi}{x} = 0$ and $\sin\frac{\pi}{x} = -1$. So at C one has $f'(x) = -\frac{3}{2}\pi$.

162 $v(x) = f(g(x)) = (x+5)^2 + 1 = x^2 + 10\,x + 26$
$w(x) = g(f(x)) = (x^2+1) + 5 = x^2 + 6$
$p(x) = f(x)g(x) = (x^2+1)(x+5) = x^3 + 5x^2 + x + 5$
$q(x) = g(x)f(x) = f(x)g(x) = p(x)$.

165 **(a)** If $f(x) = \sin ax$, then $f''(x) = -a^2 \sin ax$, so $f''(x) = -64f(x)$ holds if $a^2 = 64$, i.e. $a = \pm 8$. So $\sin 8x$ and $\sin(-8x) = -\sin 8x$ are the two solutions you find this way.
(b) $a = \pm 8$, but A and b can have any value. All functions of the form $f(x) = A \sin(8x + b)$ satisfy (†). In addition, if either $A = 0$ or $a = 0$ and $b = k\pi$, then the function $f(x)$ is always $f(x) = 0$, and also satisfies (†).

166 **(a)** $V = S^3$, so the function f for which $V(t) = f(S(t))$ is the function $f(x) = x^3$.
(b) $S'(t)$ is the rate with which Bob's side grows with time. $V'(t)$ is the rate with which the Bob's volume grows with time.

Quantity	Units
t	minutes
$S(t)$	inch
$V(t)$	inch^3
$S'(t)$	inch/minute
$V'(t)$	inch^3/minute

(c) Three versions of the same answer:
$V(t) = f(S(t))$ so the chain rule says $V'(t) = f'(S(t))S'(t)$
$V(t) = S(t)^3$ so the chain rule says $V'(t) = 3S(t)^2 S'(t)$
$V = S^3$ so the chain rule says $\dfrac{dV}{dt} = 3S^2 \dfrac{dS}{dt}$.
(d) We are given $V(t) = 8$, and $V'(t) = 2$. Since $V = S^3$ we get $S = 2$. From (c) we know $V'(t) = 3S(t)^2 S'(t)$, so $2 = 3 \cdot 2^2 \cdot S'(t)$, whence $S'(t) = \frac{1}{6}$ inch per minute.

167 $\frac{d(xy)}{dx} = 0 \implies x\frac{dy}{dx} + y = 0$. Therefore the function y satisfies $\frac{dy}{dx} = -y/x$.

168 $\frac{d\sin(xy)}{dx} = 0 \implies \cos(xy)x\frac{dy}{dx} + y\cos(xy) = 0$. As long as $\sin(xy) \neq 0$ we can divide by $\sin(xy)$, and we find that the function y satisfies $\frac{dy}{dx} = -y/x$. When $\sin(xy) = 0$ the method doesn't tell us anything.

169 Differentiate the equation defining y and you get

$$\frac{d\frac{xy}{x+y}}{dx} = \frac{\left(x\frac{dy}{dx} + y\right)(x+y) - xy(1 + \frac{dy}{dx})}{(x+y)^2} = 0.$$

Assume $x + y \neq 0$ (otherwise the defining equation $\frac{xy}{x+y} = 1$ already doesn't make sense) and solve for dy/dx (be careful, there are several cancelations). You get

$$\frac{dy}{dx} = -\frac{y^2}{x^2}$$

170 Note that this probem is the same as the previous: $\frac{xy}{x+y} = 1$ if and only if $xy = x + y$, so both equations define the same function. We should get the same answer:

$$\frac{d(xy)}{dx} = \frac{d(x+y)}{dx}$$

implies

$$x\frac{dy}{dx} + y = 1 + \frac{dy}{dx},$$

from which you can get

$$\frac{dy}{dx} = \frac{1-y}{x-1}.$$

This answer does not look like the previous answer! However, if you remember that x and y satisfy $xy = x + y$ then you can show that

$$-\frac{y^2}{x^2} = \frac{1-y}{x-1}$$

holds.

173 $\frac{dy}{dx} = 1 - \frac{1}{2(y-x)}$

175 $\frac{dy}{dx} = \frac{-1}{4y(y^2-1)}$

178 $\frac{dy}{dx} = -\frac{\cos x}{\cos y}$

179 $\frac{dy}{dx} = -\frac{y+\cos x}{5y^4+x}$

180 $\frac{dy}{dx} = -\frac{\cos^2 y}{\cos^2 x}$

181 $y = f(x)$ satisfies $y^2 = 1 - x$. Hence $\frac{dy}{dx} = -1/2y$.

182 From $y^4 = x + x^2$ you get $\frac{dy}{dx} = (1 + 2x)/4y^3$.

183 Square to get $y^2 = 1 - \sqrt{x}$ and thus $y^2 - 1 = \sqrt{x}$. Square again to get $(y^2 - 1)^2 = x$. The derivative is

$$\frac{dy}{dx} = \frac{1}{4y(y^2-1)}$$

184 y satisfies $(y^4 - x)^2 = x$. Hence

$$\frac{dy}{dx} = \frac{1}{4y^3} + \frac{1}{8y^3(y^4 - x)}$$

187 y satisfies $(y^3 - x)^2 = 2x + 1$. Hence

$$\frac{dy}{dx} = \frac{1}{3y^2} + \frac{1}{3y^2(y^3 - x)}$$

189 $\sin \arcsin x = x$ for all x with $-1 \le x \le 1$.
$\cos \arcsin x = \sqrt{1 - x^2}$ for all x with $-1 \le x \le 1$.
$\tan \arctan z = z$ fro all real numbers z.
$\arcsin(\sin\theta) = \theta$ if $-\pi/2 \le \theta \le \pi/2$
$\arctan(\tan\theta) = \theta$ if $-\pi/2 \le \theta \le \pi/2$.

190 $f'(x) = \frac{2}{\sqrt{1-4x^2}}$

191 $f'(x) = \dfrac{1}{2\sqrt{x(1-x)}}$.

192 $f'(x) = \dfrac{\cos x}{1 + \sin^2 x}$

193 $f'(x) = \dfrac{\cos \arctan x}{1 + x^2}$

194 $f'(x) = 2(\arcsin x)/\sqrt{1 - x^2}$

195 $f'(x) = \dfrac{-2\arctan x}{(1 + \arctan^2 x)(1 + x^2)}$

197 $f'(x) = \dfrac{\frac{\arcsin x}{1+x^2} - \frac{\arctan x}{\sqrt{1-x^2}}}{\arcsin^2 x}$

198 Pythagoras says that the sides $a(t)$ and $b(t)$ satisfy

$$a(t)^2 + b(t)^2 = 10^2 = 100. \qquad (*)$$

We want to find $a'(t)$. So we differentiate the relation $(*)$ to get

$$2a(t)a'(t) + 2b(t)b'(t) = 0.$$

The bottom of the pole is sliding with speed 7 feet per second, so

$$b'(t) = 7.$$

When this happens we have $b(t) = 8$, and (by Pythagoras again) $a(t) = 6$, so

$$a'(t) = -\frac{b(t)b'(t)}{a(t)} = -\frac{8 \cdot 7}{6} \text{ft/sec.}$$

199 The situation is the same as in exercise **??**. See the drawing for that problem. The angle in this problem is the angle between the pole and the wall. If we call that angle $\alpha(t)$, then $\sin\alpha(t) = b(t)/10$. Differentiate, and you find

$$\cos\alpha \frac{d\alpha}{dt} = \tfrac{1}{10}\frac{db}{dt}. \qquad (\dagger)$$

We are asked to find $d\alpha/dt$ when $\alpha = \pi/4$ and $db/dt = 10$. Equation $(\dagger)$ implies

$$\frac{d\alpha}{dt} = \frac{2}{\frac{1}{2}\sqrt{2}\cdot 10} = \tfrac{1}{5}\sqrt{2} \text{ radians/sec.}$$

200 $-5/6$ meters per second.

201 (a) Let $h(t)$ be the height of the rocket, and $d(t)$ the distance from the camera to the rocket. Pythagoras says $d(t)^2 = (4000)^2 + h(t)^2$. Differentiate this and you get

$$2d(t)d'(t) = 2h(t)h'(t). \qquad (\%)$$

We are asked to find $d'(t)$ at the moment when $h(t) = 3000$. At that moment we also have $d(t) = \sqrt{4000^2 + 3000^2} = 5000$, and hence, by (%),

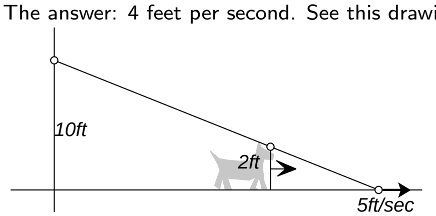
$$d'(t) = \frac{h(t)h'(t)}{d(t)} = \frac{4000 \cdot 600}{5000} = 480\text{ft/sec}$$

(b) Call the angle $\theta(t)$. Then $\tan\theta(t) = h(t)/4000$, and thus

$$\frac{1}{\cos^2\theta}\theta'(t) = h'(t)/4000 \qquad (\#)$$

Since $h'(t) = 600$ we get $\theta'(t) = \frac{3}{20}\cos^2\theta(t)$. When the rocket has reached height 3000 we have $d(t) = 5000$ and thus $\cos\theta = 4/5$. Therefore the angle θ is increasing at a rate of $\theta'(t) = (\frac{4}{5})^2\frac{3}{20} = \frac{12}{125}$ radians per second.

202 The answer: 4 feet per second. See this drawing



203 Let one of the two equal sides be the base of the triangle. Then the height of the triangle is $2\sin\theta$, and its area is $A(t) = 2\sin\theta(t)$. Therefore

$$\frac{dA}{dt} = 2\cos\theta(t)\frac{d\theta}{dt}.$$

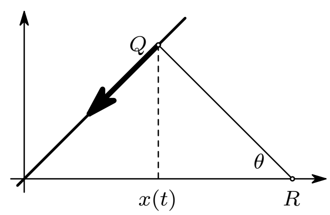
At the moment that $\theta = 60^\circ$ you get $A'(t) = 2 \times \frac{1}{2}\theta'(t)$, and therefore $\theta(t) = A'(t) = 1$ radian per second.

204 The y coordinate of P is always 10. Let $x(t)$ be the x coordinate of the point P at time t. The $\frac{10}{x(t)} = \tan\theta(t)$, or $10 = x(t)\tan\theta(t)$.
(b) When $\theta = \pi/3$ one has $x(t) = \frac{10}{3}\sqrt{3}$.
(c) Differentiate $10 = x(t)\tan\theta(t)$, to get

$$x'(t)\tan\theta(t) + x(t)\theta'(t)/\cos^2\theta(t) = 0$$

Substitute $\theta = \pi/3$ (radians) and $x' = -3$ (feet per second), and you get $\theta'(t) = \frac{9}{40}$ radians per second.

205 The situation is as follows:



(a) The distance from Q to the origin decreases at 3 m/sec.
(b) Let $x(t)$ be the x coordinate of the point Q. It is then also the y coordinate because Q lies on the line $y = x$. The distance from Q to the origin is $OQ = \sqrt{2}x(t)$. Therefore $x'(t) = -3/\sqrt{2} = -\frac{3}{2}\sqrt{2}$ meters per second. If the distance from Q to R is $d(t)$, then

$$d(t)^2 = (2 - x(t))^2 + x(t)^2 = 4 - 2x(t) + 2x(t)^2.$$

Differentiate:

$$2d(t)d'(t) = \big(-2 + 4x(t)\big)x'(t).$$

At the given moment we have $x = 2$, $x' = -\frac{3}{2}\sqrt{2}$ and $d = 2\sqrt{2}$, so

$$d'(t) = -\frac{3 \cdot \frac{3}{2}\sqrt{2}}{2\sqrt{2}} = -\frac{9}{4}.$$

(c) Let $\theta(t)$ be the angle $\angle ORQ$. Then

$$\tan\theta = \frac{x}{2 - x}.$$

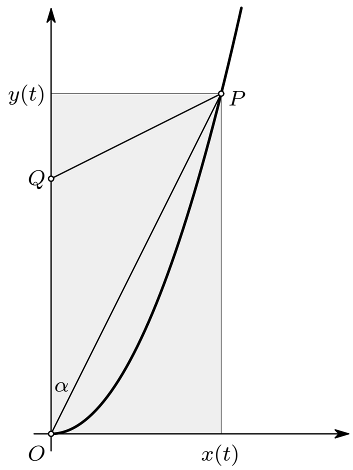
Differentiate, to get

$$\frac{1}{\cos^2\theta}\theta'(t) = \frac{x'(t)}{(2 - x(t))^2}.$$

When $x(t) = 1$ you have $\cos\theta = \frac{1}{2}\sqrt{2}$, and therefore

$$\theta'(t) = \frac{1}{2}\,\frac{-\frac{3}{2}\sqrt{2}}{1^2} = -\frac{3}{4}\sqrt{2}.$$

206 Here's a drawing:



(a) If $d(t)$ is the distance from P to the origin, then

$$d(t)^2 = x(t)^2 + y(t)^2 = x(t)^2 + x(t)^4.$$

Hence

$$2d(t)d'(t) = \big(2x(t) + 4x(t)^3\big)x'(t).$$

When P is $(3, 9)$, then $x = 3$, $d = \sqrt{90} = 3\sqrt{10}$, and $x' = 2$, so we get

$$d' = \frac{2 \cdot 6 + 4 \cdot 3^3}{3\sqrt{10}} = \cdots$$

(b) The area of the rectangle it $A(t) = x(t)y(t) = x(t)^3$. Hence $A'(t) = 3x(t)^2x'(t) = 3 \cdot 3^2 \cdot 2 = 54$ (square inch per second).
(c) The slope $m(t)$ of the tangent at P is $m(t) = 2x(t)$, so its rate of change is $m'(t) = 2x'(t) = 4$.
(d) The angle $\angle QOP$ is $\alpha(t)$ in the drawing above. One has $\tan\alpha(t) = x(t)/y(t) = 1/x(t)$, so, if you differentiate this relation, you get

$$\frac{\alpha'(t)}{\cos^2\alpha} = -\frac{x'(t)}{x(t)^2}$$

whence $\alpha'(t) = -\cos^2\alpha(t)x'(t)/x(t)^2$.
When $x = 3$ and $x' = 2$ this implies $\cos\alpha = 3/\sqrt{10}$ and thus $\alpha' = -\frac{9}{10}\frac{2}{9} = -\frac{1}{5}$.

208 At $x = 3$.

209 At $x = a/2$.

210 At $x = a + 2a^3$.

211 At $x = a + \frac{1}{2}$.

215 False. If you try to solve $f(x) = 0$, then you get the equation $\frac{x^2+|x|}{x} = 0$. If $x \neq 0$ then this is the same as $x^2 + |x| = 0$, which has no solutions (both terms are positive when $x \neq 0$). If $x = 0$ then $f(x)$ isn't even defined. So there is no solution to $f(x) = 0$.
This doesn't contradict the IVT, because the function isn't continuous, in fact it isn't even defined at $x = 0$, so the IVT doesn't have to apply.

223 Not necessarily true, and therefore false. Consider the example $f(x) = x^4$, and see the next problem.

224 An inflection point is a point on the graph of a function where the second derivative changes its sign. At such a point you must have $f''(x) = 0$, but by itself that it is no enough.

227 The first is possible, e.g. $f(x) = x$ satisfies $f'(x) > 0$ and $f''(x) = 0$ for all x.
The second is impossible, since f'' is the derivative of f', so $f'(x) = 0$ for all x implies that $f''(x) = 0$ for all x.

228 $y = 0$ at $x = -1, 0, 0$. Only sign change at $x = -1$, not at $x = 0$.
$x = 0$ loc min; $x = -\frac{4}{3}$ loc max; $x = -2/3$ inflection point. No global max or min.

229 zero at $x = 0, 4$; sign change at $x = 4$; loc min at $x = \frac{8}{3}$; loc max at $x = 0$; inflection point at $x = 4/3$. No global max or min.

230 sign changes at $x = 0, -3$; global min at $x = -3/4^{1/3}$; no inflection poitns, the graph is convex.

231 mirror image of previous problem.

232 $x^4 + 2x^2 - 3 = \big(x^2 - 1\big)\big(x^2 + 3\big)$ so sign changes at $x = \pm 1$. Global min at $x = 0$; graph is convex, no inflection points.

233 Sign changes at $\pm 2, \pm 1$; **two** global minima, at $\pm\sqrt{5/2}$; one local max at x=0; two inflection points, at $x = \pm\sqrt{5/6}$.

234 Sign change at $x = 0$; function is always increasing so no stationary points; inflection point at $x = 0$.

235 sign change at $x = 0, \pm 2$; loc max at $x = 2/5^{1/4}$; loc min at $x = -2/5^{1/4}$. inflection point at $x = 0$.

236 Function not defined at $x = -1$. For $x > -1$ sign change at $x = 0$, no stationary points, no inflection points (graph is concave). Horizontal asymptote $\lim_{x\to\infty} f(x) = 1$.
For $x < -1$ no sign change , function is increasing and convex, horizontal asymptote with $\lim_{x\to-\infty} f(x) = 1$.

237 global max (min) at $x = 1$ (x=-1), inflection points at $x = \pm\sqrt{3}$; horizontal asymptotes $\lim_{x\to\pm\infty} f(x) = 0$.

238 $y = 0$ at $x = 0$ but no sign changes anywhere; $x = 0$ is a global min; there's no local or global max; two inflection points at $x = \pm\frac{1}{3}\sqrt{3}$; horizontal asymptotes at height $y = 1$.

239 Not defined at $x = -1$. For $x > -1$ the graph is convex and has a minimum at $x = -1 + \sqrt{2}$; for $x < -1$ the graph is concave with a maximum at $x = -1 - \sqrt{2}$. No horizontal aymptotes.

240 Not def'd at $x = 0$. No sign changes (except at $x = 0$). For $x > 0$ convex with minimum at $x = 1$, for $x < 0$ concave with maximum at $x = -1$.

241 Not def'd at $x = 0$. Sign changes at $x = \pm 1$ and also at $x = 0$. No stationary points. Both branches ($x > 0$ and $x < 0$) are increasing. Non inflection points, no horizontal asymptotes.

242 Zero at $x = 0, -1$ sign only changes at -1; loc min at $x = -\frac{1}{3}$; loc max at $x = -1$. Inflection point at $x = -2/3$.

243 Changes sign at $x = -1 \pm \sqrt{2}$ and $x = 0$; loc min at $(-2 + \sqrt{7})/3$, loc max at $(-2 - \sqrt{7})/3$; inflection point at $x = -\frac{2}{3}$.

244 Factor $y = x^4 - x^3 - x = x(x^3 - x^2 - 1)$. One zero is obvious, namely at $x = 0$. For the other(s) you must solve $x^3 - x^2 - 1 = 0$ which is beyond what's expected in this course.
The derivative is $y' = 4x^3 - 3x^2 - 1$. A cubic function whose coefficients add up to 0 so $x = 1$ is a root, and you can factor $y' = 4x^3 - 3x^2 - 1 = (x-1)(4x^2+x+1)$ from which you see that $x = 1$ is the only root. So: one stationary point at $x = 1$, which is a global minimum
The second derivative is $y'' = 12x^2 - 6x$; there are two inflection points, at $x = \frac{1}{2}$ and at $x = 0$.

245 Again one obvious solution to $y = 0$, namely $x = 0$. The other require solving a cubic equation.
The derivative is $y' = 4x^3 - 6x^2 + 2$ which is also cubic, but the coefficients add up to 0, so $x = 1$ is a root. You can then factor $y' = 4x^3 - 6x^2 + 2 = (x-1)(4x^2 - 2x - 2)$. There are three stationary points: local minima at $x = 1$, $x = -\frac{1}{4} - \frac{1}{2}\sqrt{3}$, local max at $x = -\frac{1}{4} + \frac{1}{2}\sqrt{3}$. one of the two loc min is a global minimum.

246 Global min at $x = 0$, no other stationary points; function is convex, no inflection points. No horizontal asymptotes.

247 The graph is the upper half of the unit circle.

248 Always positive, so no sign changes; global minimum at $x = 0$, no other stationary points; two inflection points at $\pm\sqrt{2}$. No horizontal asymptotes since $\lim_{x\to\pm\infty} \sqrt[4]{1+x^2} = \infty(\text{DNE})$.

249 Always positive hence no sign changes; global max at $x = 0$, no other stationary points; two inflection points at $x = \pm\sqrt[4]{3/5}$; second derivative also vanishes at $x = 0$ but this is not an inflection point.

251 Zeroes at $x = 3\pi/4,\ 7\pi/4$. Absolute max at $x = \pi/4$, abs min at $x = 5\pi/4$. Inflection points and zeroes coincide. Note that $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$.

252 Zeroes at $x = 0, \pi, 3\pi/2$ but no sign change at $3\pi/2$. Global max at $x = \pi/2$, local max at $x = 3\pi/2$, global min at $x = 7\pi/6, 11\pi/6$.

269 If the length of one side is x and the other y, then the perimeter is $2x + 2y = 1$, so $y = \frac{1}{2} - x$. Thus the area enclosed is $A(x) = x(\frac{1}{2} - x)$, and we're only interested in values of x between 0 and $\frac{1}{2}$.
The maximal area occurs when $x = \frac{1}{4}$ (and it is $A(1/4) = 1/16$.) The minimal area occurs when either $x = 0$ or $x = 1/2$. In either case the "rectangle" is a line segment of length $\frac{1}{2}$ and width 0, or the other way around. So the minimal area is 0.

270 If the sides are x and y, then the area is $xy = 100$, so $y = 1/x$. Therefore the height plus twice the width is $f(x) = x + 2y = x + 2/x$. This is extremal when $f'(x) = 0$, i.e. when $f'(x) = 1 - 2/x^2 = 0$. This happens for $x = \sqrt{2}$.

271 Perimeter is $2R + R\theta = 1$ (given), so if you choose the angle to be θ then the radius is $R = 1/(2+\theta)$. The area is then $A(\theta) = \theta R^2 = \theta/(2+\theta)^2$, which is maximal when $\theta = 2$ (radians). The smallest area arises when you choose $\theta = 0$. Choosing $\theta \geq 2\pi$ doesn't make sense (why? Draw the corresponding wedge!)
You could also say that for any given radius $R > 0$ "perimeter $= 1$" implies that one has $\theta = (1/R) - 2$. Hence the area will be $A(R) = \theta R^2 = R^2\big((1/R) - 2)\big) = R - 2R^2$. Thus the area is maximal when $R = \frac{1}{4}$, and hence $\theta = 2$radians. Again we note that this answer is reasonable because values of $\theta > 2\pi$ don't make sense, but $\theta = 2$ does.

272 **(a)** The intensity at x is a function of x. Let's call it $I(x)$. Then at x the distance to the big light is x, and the distance to the smaller light is $1000 - x$. Therefore

$$I(x) = \frac{1000}{x^2} + \frac{125}{(1000-x)^2}$$

(b) Find the minimum of $I(x)$ for $0 < x < 1000$.

$$I'(x) = -2000x^{-3} + 250(1000-x)^{-3}.$$

$I'(x) = 0$ has one solution, namely, $x = \frac{1000}{3}$. By looking at the signs of $I'(x)$ you see that $I(x)$ must have a minimum. If you don't like looking at signs, you could instead look at the second derivative

$$I''(x) = 6000x^{-4} + 750(1000-x)^{-4}$$

which is always positive.

273 $r = \sqrt{50/3\pi}$, $h = 100/(3\pi r) = 100/\sqrt{150\pi}$.

284 $dy/dx = e^x - 2e^{-2x}$. Local min at $x = \frac{1}{3}\ln 2$.
$d^2y/dx^2 = e^x + 4e^{-2x} > 0$ always, so the function is convex.
$\lim_{x\to\pm\infty} y = \infty$, no asymptotes.

285 $dy/dx = 3e^{3x} - 4e^x$. Local min at $x = \frac{1}{2}\ln\frac{4}{3}$.
$d^2y/dx^2 = 9e^{3x} - 4e^x$ changes sign when $e^{2x} = \frac{4}{9}$, i.e. at $x = \frac{1}{2}\ln\frac{4}{9} = \ln\frac{2}{3} = \ln 2 - \ln 3$. Inflection point at $x = \ln 2 - \ln 3$.
$\lim_{x\to-\infty} f(x) = 0$ so negative x axis is a horizontal asymptote.
$\lim_{x\to\infty} f(x) = \infty$... no asymptote there.

331 Choosing left endpoints for the c's gives you

$$f(0)\frac{1}{3} + f(\frac{1}{3})\frac{2}{3} + f(1)\frac{1}{2} + f(\frac{3}{2})\frac{1}{2} = \cdots$$

Choosing right endpoints gives

$$f(\frac{1}{3})\frac{1}{3} + f(1)\frac{2}{3} + f(\frac{3}{2})\frac{1}{2} + f(2)\frac{1}{2} = \cdots$$

332 The Riemann-sum is the total area of the rectangles, so to get the smallest Riemann-sum you must make the rectangles as small as possible. You can't change their

widths, but you can change their heights by changing the c_i. To get the smallest area we make the heights as small as possible. Since f appears to be decreasing, the heights $f(c_i)$ will be smallest when c_i is as large as possible. So we choose the intermediate points c_i all the way to the right of the interval $x_{i-1} \le c_i \le x_i$, i.e. $c_1 = x_1$, $c_2 = x_2$, $c_3 = x_3$, $c_4 = x_4$, $c_5 = x_5$, $c_6 = b$,
To get the *largest* Riemann-sums you choose $c_1 = a$, $c_2 = x_1, \ldots, c_6 = x_5$.

373 **(a)** The first derivative of $\operatorname{erf}(x)$ is, by definition

$$\operatorname{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2},$$

so you get the second derivative by differentiating this:

$$\operatorname{erf}''(x) = \frac{-4x}{\sqrt{\pi}} e^{-x^2}.$$

This is negative when $x > 0$, and positive when $x < 0$ so the graph of $\operatorname{erf}(x)$ has an inflection point at $x = 0$.
(b) Wikipedia is not wrong. Let's figure out what sign $\operatorname{erf}(-1)$ has (for instance). By definition you have

$$\operatorname{erf}(-1) = \frac{2}{\sqrt{\pi}} \int_0^{-1} e^{-t^2} dt.$$

Note that in this integral the upper bound (-1) is less than the lower bound (0). To fix that we switch the upper and lower integration bounds, which introduces a minus sign:

$$\operatorname{erf}(-1) = -\frac{2}{\sqrt{\pi}} \int_{-1}^{0} e^{-t^2} dt.$$

The integral we have here is positive because it's an integral of a positive function from a smaller number to a larger number, i.e. it is of the form $\int_a^b f(x)dx$ with $f(x) \ge 0$ and with $a < b$.
With the minus sign that makes $\operatorname{erf}(-1)$ negative.

487 The answer is $\pi/6$.
To get this using the integral you use formula (**??**) with $f(x) = \sqrt{1-x^2}$. You get $f'(x) = -x/\sqrt{1-x^2}$, so

$$\sqrt{1 + f'(x)^2} = \frac{1}{\sqrt{1-x^2}}.$$

The integral of that is $\arcsin x (+C)$, so the answer is $\arcsin \frac{1}{2} - \arcsin 0 = \pi/6$.