Math 521 **Midterm 2** **Fall 2012**

NAME:

Grading Table

Question	Possible Points	Points Earned
1	10	
2	10	
3	10	
4	15	
Total	45	

(1) (10 points) For each of the following, give an example if one exists. If there is none, state that there is no example. You do not need to give any justification.

(a) A continous map from $[0,1]$ onto $(0,1)$.
Answer: None exists: $[0,1]$ is compact and the continuous image of a compact set is compact.

(b) A continous function and a connected set A so that $f^{\text{pre}}(A)$ is disconnected.
Answer: $f(x) = x^2$ and $A = [1,2]$.

(c) A continous function $f : (0,1) \to \mathbb{R}$ with unbounded image.
Answer: $f(x) = \frac{1}{x}$

(d) A uniformly continous function $f : (0,1) \to \mathbb{R}$ with unbounded image.
Answer: None exists. If f is uniformly continuous on $(0,1)$ it can be extended (uniquely) to a continuous function on $[0,1]$, which must be bounded.

(e) A nested sequence $A_0 \supseteq A_1 \supseteq A_2 \ldots$ of closed subsets of $\mathbb{R}^2$ with empty intersection.
Answer: $A_i = \{(x,y) \mid x \geq i\}$

(2) (10 points) If $A \subseteq \mathbb{R}^n$ is non-empty and closed and if x is any given point in $\mathbb{R}^n$, prove that there is a point $y \in A$ of minimum distance from x. In other words,

$$\exists y \in A \text{ such that } \forall z \in A\,(d(x,z) \geq d(x,y))$$

Answer: This problem (as far as I can see) really needs the idea of compactness. Let b be any point in A. Let's look at the set $A' = \{p \in A \mid d(p,x) \leq d(b,x)\} = A \cap B_{\leq d(b,x)}(x)$. A' is closed and bounded in $\mathbb{R}^n$, so compact. Thus there is a point of minimal distance from x in A'. That's an a so that whenever $z \in A'$, $d(z,x) \geq d(a,x)$. But if $z \in A \setminus A'$, then $d(z,x) > d(b,x) \geq d(a,x)$. Thus this a is of minimal distance from x.

(3) (10 points) Show that the following two conditions are equivalent for a function $f : \mathbb{R} \to \mathbb{R}$

(a) f is uniformly continuous.

(b) For any pair of (not necessarily convergent) sequences (x_n) and (y_n) in $\mathbb{R}$, if $(x_n - y_n)_{n\in\mathbb{N}}$ converges to 0 then $(f(x_n) - f(y_n))_{n\in\mathbb{N}}$ converges to 0.

Answer: $\downarrow$: For any $\epsilon > 0$, we want to find an N so that $n > N$ implies $|f(x_n) - f(y_n)| < \epsilon$. By uniform continuity, there is a δ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. As $(x_n - y_n)$ converges to 0, there is an N so that $|x_n - y_n| < \delta$ whenever $n > N$. Putting these two together, we see that $n > N$ implies $|x_n - y_n| < \delta$ which in turn implies $|f(x) - f(y)| < \epsilon$. Thus this N works.

$\uparrow$: We'll prove this contrapositively. Suppose f is not uniformly continuous. That is, there is an $\epsilon > 0$ so that for any $\delta > 0$, there are some points x, y where $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon$. For each n, let x_n and y_n be so that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$. Then this is a pair of sequences where $(x_n - y_n)$ converges to 0, but $(f(x_n) - f(y_n))$ does not converge to 0.

(4) (15 points)

(a) A connected component of a set A is a maximal subset $B \subseteq A$ which is connected. That is, B is connected and no C properly containing B is connected. Give an example of a subset of $\mathbb{R}$ with uncountably many connected components.

Answer: The cantor set, or $\mathbb{R} \setminus \mathbb{Q}$. The connected components are just the sets of size 1. There are uncountably many points in C and in $\mathbb{R} \setminus \mathbb{Q}$.

(b) Suppose that X is connected and $f, g : X \to [0, 1]$ are two continuous functions and f is onto. Show that there is some x in X so that $f(x) = g(x)$.

Answer: Let a be so that $f(a) = 0$ and b be so that $f(b) = 1$ (these exist, since f is onto). Consider the function $h(x) = f(x) - g(x)$. $h(a) \leq 0$ while $h(b) \geq 0$. Since f and g are continuous, so is h. Since X is connected, the IVT says that for some point c, $h(c) = 0$. But then $f(c) = g(c)$, as required.